SEQUENTIAL DECISIONS WITH OUTCOME UNCERTAINTY

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CHAPTER I

INTRODUCTION

There are a number of practical problems where a sequence of decisions are to be made as to which action is to be taken on some known system and where the outcomes of previous actions on that system are uncertain. In many such problems it is possible, between decisions, to obtain, in one of several alternative ways, some information which reduces the uncertainty as to the outcomes of previous actions. In problems of this type, it is usually desired to make the decisions and obtain information in some optimal way.

Stochastic control problems provide an abundance of examples of the type indicated briefly above. For instance, suppose there is a control system whose dynamics are specified by the first order linear difference equation,

$$X_j = X_{j-1} + d_j \qquad j = 1,2,\ldots,n \qquad .$$

X_j represents the unobservable state of the system at time j and d_j represents the control, or decision, at time j. X_0 is not known, but is a random variable with known distribution. For each j, the control is to be some real number. Between the time j and the time j+1, information as to the value of the state X_j may be obtained in one of two ways which are denoted α_j and β_j. The choice α_j has associated with it a cost $C(\alpha_j)$ and results in an observation of a random variable I^{α_j}.

Similarly, the choice β_j has associated with it a cost $C(\beta_j)$ and results in an observation of a random variable I^{β_j}. The sequence of events, as they occur, is indicated by the tree diagram shown in Figure 1. Notice that x's represent places where the decision maker makes a choice and o's represent places where chance determines an outcome.

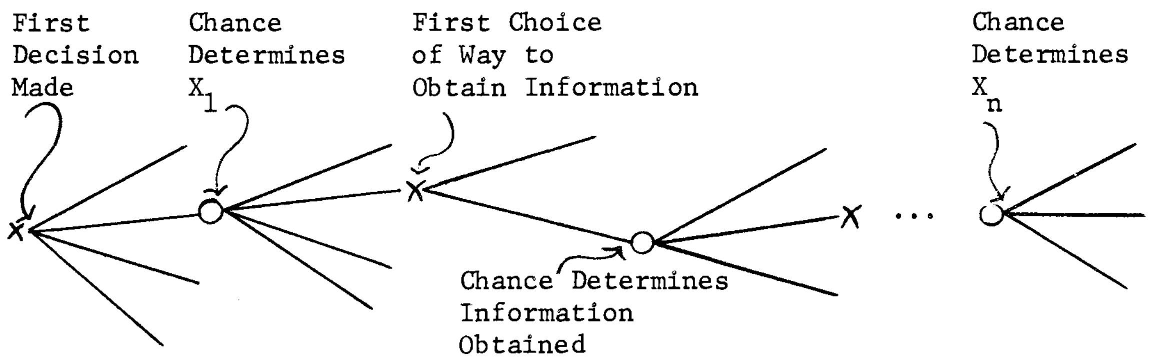

Figure 1. Tree Diagram of the Stochastic Control Example

The desired state of the system is specified for each time, $j = 1,2,\ldots,n$, and the problem is to choose the decisions and ways of obtaining information so as to minimize the expectation of the sum of the mean square errors and costs of obtaining information. This problem will be solved in a later chapter.

This investigation is centered around the class, or type, of sequential decision theory problem which was roughly described in the first paragraph of this chapter. In the next chapter a precise formulation of the class of problems investigated is given. The formulation is followed by a discussion of a general method of solution. Also given in Chapter II are some theorems which may be used in some cases to greatly reduce the computational effort required in obtaining a solution to a particular problem. In generalized form, the class of problems considered contain, as special cases, a number of practical applications. Applications to control, weapon analysis, and systems engineering problems are used, in Chapters III, IV, and V to illustrate the results obtained in Chapter II.

There are a number of statistical decision theory problems solved in the literature which are similar, in one respect or another, to the class of problems considered here. See for example DeGroot (1970), Blackwell and Girshick (1954), and Raiffa and Schlaifer (1961). The distinguishing characteristic of the class of problems being considered here is the sequence of decisions resulting in at least partially unobservable outcomes combined with a choice of ways to obtain information between decisions.

CHAPTER II

FORMULATION, SOLUTION, AND SOME SIMPLIFYING THEOREMS

Formulation

The class of problems to be investigated are finite stage, discrete time, sequential decision theory problems. For each integer $j = 1,2,\ldots,n$ a decision, d_j, will be made and these decisions will be made in order of increasing subscript. For each j, the decision d_j must be selected from a given set, λ_j. Once d_j is chosen and some corresponding action is taken, a randomized outcome, x_j, occurs. The values which x_j may take will be denoted by the set Ω_{X_j}. For $j = 1,2, \ldots,n$, the ordered collections of the first j decisions and the first j outcomes are denoted by $\underline{d}_j$ and $\underline{x}_j$ respectively. That is, $\underline{d}_j = (d_1,d_2,\ldots,d_j)$ and $\underline{x}_j = (x_1,x_2,\ldots,x_j)$. At the time for the $(j+1)^{th}$ decision, $\underline{d}_j$ is known, but even though the collection of outcomes, $\underline{x}_j$, has occurred, it is in general not known except possibly in an approximate sense.

Between the time of the outcome, x_j, $j = 1,2,\ldots,n-1$ and the time for the decision, d_{j+1}, information may be sought as to the true value of $\underline{x}_j$ by selecting an experiment, k_j. For each j, the experiment k_j must be selected from a given set ω_j. The cost associated with this experimentation is $C(k_j)$ where C is a given nonnegative cost function. As a result of the particular choice $k_j \varepsilon \omega_j$, a sample of the random variable(s), I^{k_j}, is obtained. The particular sample outcome, i^{k_j}, is contained in a set denoted Ω_{k_j}. For each $j = 1,2,\ldots,n-1$, $*j$, the null experiment or the experiment resulting in an observation of a degenerate random variable, I^{*j}, is an element of ω_j. Further, for each $j = 1,2,\ldots,n-1$, $C(*j) = 0$.

The following two probability density functions are given or derivable from information given in the particular problem:

$$f_{\underline{X}_n|\underline{D}_n}(\underline{x}_n|\underline{d}_n) \ \forall\ \underline{d}_n \in \lambda_1 \times \lambda_2 \times \ldots \times \lambda_n = \underline{\lambda}_n$$

and

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n}(i^{\underline{k}_{n-1}}|\underline{d}_n,\underline{x}_n) \ \forall\ \underline{k}_{n-1} \in \omega_1 \times \omega_2 \times \ldots \times \omega_{n-1} = \underline{\omega}_{n-1}$$

$$\underline{d}_n \in \underline{\lambda}_n \text{ and}$$

$$\underline{x}_n \in \Omega_{X_1} \times \Omega_{X_2} \times \ldots \times \Omega_{X_n} = \Omega_{\underline{X}_n}$$

where

$\underline{k}_j = (k_1,k_2,\ldots,k_j)$, $j = 1,2,\ldots,n-1$

$I^{\underline{k}_j} = (I^{k_1},I^{k_2},\ldots,I^{k_j})$, $j = 1,2,\ldots,n-1$, and

x has been used to indicate the cartesian product.

The symbol $\Omega_{\underline{k}_j}$ will be used to indicate $\Omega_{k_1} \times \Omega_{k_2} \times \ldots \times \Omega_{k_j}$, $j = 1,2,\ldots,n-1$.

At this point a comment should be made in regard to notation. For the sake of efficiency, when there appears to be no sacrifice of clarity and when arguments are to be taken as lower case versions of the indicated variables, all or part of the arguments of probability density functions will be omitted. For example, $f_{\underline{X}_n|\underline{D}_n}(|)$ might be written for $f_{\underline{X}_n|\underline{D}_n}(\underline{x}_n|\underline{d}_n)$.

Marginal conditional density functions for $\underline{X}_j$ and $I^{\underline{k}_j}$ may be derived from the given density functions. As a characteristic of the general class of problems being considered it is assumed that the conditional density of outcomes, $\underline{X}_j$, is independent of decisions which have not occurred at the time X_j occurs. Also, the conditional density of observations, $I^{\underline{k}_j}$, is independent of decisions and outcomes which have not occurred at the time of the observation. Stated in equation form,

$$f_{\underline{X}_j|\underline{D}_p} = f_{\underline{X}_j|\underline{D}_j}, \qquad 1 \leq j < p \leq n$$

and

$$f_{I^{\underline{k}_j}|\underline{D}_p,\underline{X}_q} = f_{I^{\underline{k}_j}|\underline{D}_j,\underline{X}_j}, \qquad \begin{array}{l} 1 \leq j < p \leq n \\ 1 \leq j < q \leq n \end{array} \quad .$$

There is given a set of utility functions, $V_1(\underline{x}_1,\underline{d}_1)$, $V_2(\underline{x}_2,\underline{d}_2)$, $\ldots,V_n(\underline{x}_n,\underline{d}_n)$, which represent the decision makers preferences in the sense that he would like to choose $\underline{d}_n$ and $\underline{k}_{n-1}$ in such a way as to maximize the expectation of

$$\sum_{j=1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=1}^{n-1} C(k_j) \quad .$$

When it proves expedient, the following notation is used:

$$V(\underline{x}_n,\underline{d}_n) \text{ to indicate } \sum_{j=1}^{n} V_j(\underline{x}_j,\underline{d}_j)$$

and

$$C(\underline{k}_{n-1}) \text{ to indicate } \sum_{j=1}^{n-1} C(k_j) \quad .$$

The problem is to determine the sequential rules for the decisions and experiment choices so as to maximize the expectation of V - C.

General Solution

The solution to the problem which has been formulated may be obtained by the method of backward induction (DeGroot, 1970). This method consists of working backward from the n^{th} decision, establishing optimal rules for each decision and experiment choice in terms of what is known at the time that decision or experiment choice is to be made.

Accordingly, the conditional expectation of V - C is computed for each condition $\underline{d}_n \varepsilon \underline{\lambda}_n$, $\underline{k}_{n-1} \varepsilon \underline{\omega}_{n-1}$ and $i^{\underline{k}_{n-1}} \varepsilon \Omega_{\underline{k}_{n-1}}$. The required expectation is given by:

$$\int_{\Omega_{\underline{X}_n}} V(\underline{x}_n,\underline{d}_n) f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}} d\underline{x}_n - C(\underline{k}_{n-1})$$

$$= \sum_{j=1}^{n} \int_{\Omega_{\underline{X}_j}} V_j(\underline{x}_j,\underline{d}_j) f(|)_{\underline{X}_j|\underline{D}_n, I^{\underline{k}_{n-1}}} d\underline{x}_j - C(\underline{k}_{n-1})$$

where by Bayes' formula,

$$f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}} = \frac{f(|)_{\underline{X}_n|\underline{D}_n} \; f_{I^{\underline{k}_{n-1}}|\underline{X}_n,\underline{D}_n}(|)}{\int_{\Omega_{\underline{X}_n}} f(|)_{\underline{X}_n|\underline{D}_n} \; f_{I^{\underline{k}_{n-1}}|\underline{X}_n,\underline{D}_n}(|) \, d\underline{x}_n} .$$

The next step is to determine, by differentiation or other means, the optimal rule, $\tilde{d}_n(\underline{d}_{n-1},\underline{k}_{n-1},i^{\underline{k}_{n-1}})$, for the n^{th} decision, which as indicated, is a function of the state of knowledge at the time the n^{th}

decision must be made. This optimal rule must maximize the conditional expectation of V - C given $\underline{d}_n$, $\underline{k}_{n-1}$, and $I^{\underline{k}_{n-1}} = i^{\underline{k}_{n-1}}$ as computed above. Notice it has been assumed that such a rule exists. Actually, there is no guarantee at this point that for some $\underline{d}_{n-1}$, $\underline{k}_{n-1}$ and $i^{\underline{k}_{n-1}}$ there are not two or more values of d_n which yield the same maximum; however, in such cases an arbitrary choice would be made among the alternate decisions. Further, it is possible that for some $\underline{d}_{n-1}$, $\underline{k}_{n-1}$, and $i^{\underline{k}_{n-1}}$ there is no $d_n \varepsilon \lambda_n$ which maximizes the conditional expectation of V - C. In other words, it may be that for any $d_n \varepsilon \lambda_n$ there exists a $d'_n \varepsilon \lambda_n$ such that the expectation with d_n is less than the expectation with d'_n. In such a situation, $\tilde{d}_n$ would be chosen such that the expectation with $\tilde{d}_n$ is sufficiently close to the supremum of the expectation over all $d_n \varepsilon \lambda_n$. Since such difficulties seem to be of little practical difficulty, we will henceforth tacitly assume that they do not exist and that optimal rules do exist and can be found.

Next, substitution of the optimal rule, $\tilde{d}_n$, into the conditional expectation of V - C given $\underline{d}_n$, $\underline{k}_{n-1}$ and $I^{\underline{k}_{n-1}} = i^{\underline{k}_{n-1}}$ yields the maximum expectation of V - C given $\underline{d}_{n-1}$, $\underline{k}_{n-1}$, and $I^{\underline{k}_{n-1}} = i^{\underline{k}_{n-1}}$. This maximum expectation can be written

$$\max_{d_n} \left[\int_{\Omega_{\underline{X}_n}} V(\underline{x}_n, \underline{d}_n) f(\,|\,)_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-1}}} \, d\underline{x}_n \right] - C(\underline{k}_{n-1})$$

$$= \sum_{j=1}^{n-1} \int_{\Omega_{\underline{X}_j}} V_j(\underline{x}_j, \underline{d}_j) f(\,|\,)_{\underline{X}_j | \underline{D}_{n-1}, I^{\underline{k}_{n-1}}} \, d\underline{x}_j$$

$$+ \max_{d_n} \left[\int_{\Omega_{\underline{X}_n}} V_n(\underline{x}_n, \underline{d}_n) f(\,|\,)_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-1}}} \, d\underline{x}_n \right] - C(\underline{k}_{n-1}) \quad .$$

The next step is to determine the optimum rule for the $(n-1)^{th}$ experiment choice, $\tilde{k}_{n-1}(\underline{d}_{n-1},\underline{k}_{n-2},i^{\underline{k}_{n-2}})$, which maximizes the conditional expectation of the maximum expectation of V - C given $\underline{d}_{n-1}$, $\underline{k}_{n-1}$, and $I^{\underline{k}_{n-1}} = i^{\underline{k}_{n-1}}$ given $\underline{d}_{n-1}$, $\underline{k}_{n-2}$, and $I^{\underline{k}_{n-2}} = i^{\underline{k}_{n-2}}$. The required expectation is given by:

$$\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,) \max_{d_n}\left[\int_{\Omega_{\underline{X}_n}} V(\underline{x}_n,\underline{d}_n) f_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{n-1}}}(\,|\,)\, d\underline{x}_n\right] di^{k_{n-1}} - C(\underline{k}_{n-1})$$

where

$$f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,) = \int_{\Omega_{\underline{X}_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}},\underline{X}_{n-1}}(\,|\,)\, f_{\underline{X}_{n-1}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)\, d\underline{x}_{n-1} \,.$$

Substitution of the rule, $\tilde{k}_{n-1}$, into the above expectation yields the maximum expectation of V - C given $\underline{d}_{n-1}$, $\underline{k}_{n-2}$, and $I^{\underline{k}_{n-2}} = i^{\underline{k}_{n-2}}$. This maximum expectation can be written

$$\max_{k_{n-1}}\left[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)\right.$$

$$\left.\max_{d_n}\left[\int_{\Omega_{\underline{X}_n}} V(\underline{x}_n,\underline{d}_n) f_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{n-1}}}(\,|\,)\, d\underline{x}_n\right] di^{k_{n-1}} - C(k_{n-1})\right] - \sum_{j=1}^{n-2} C(k_j)$$

$$= \sum_{j=1}^{n-1} \int_{\Omega_{\underline{X}_j}} V_j(\underline{x}_j,\underline{d}_j) f_{\underline{X}_j|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)\, d\underline{x}_j + \max_{k_{n-1}}\left[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)\right.$$

$$\left.\max_{d_n}\left[\int_{\Omega_{\underline{X}_n}} V_n(\underline{x}_n,\underline{d}_n) f_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{n-1}}}(\,|\,)\, d\underline{x}_n\right] di^{k_{n-1}} - C(k_{n-1})\right] - \sum_{j=1}^{n-2} C(k_j) \quad .$$

Continuing in this fashion, the complete set of optimal decision and experiment rules can be obtained. The listing given below indicates the variables or the state of knowledge the successive rules are dependent upon.

$$\tilde{d}_1$$

$$\tilde{k}_1(\underline{d}_1)$$

$$\tilde{d}_2(\underline{d}_1,\underline{k}_1,i^{\underline{k}_1})$$

$$\tilde{k}_2(\underline{d}_2,\underline{k}_1,i^{\underline{k}_1})$$

$$\tilde{d}_3(\underline{d}_2,\underline{k}_2,i^{\underline{k}_2})$$

$$\vdots$$

$$\tilde{k}_{n-1}(\underline{d}_{n-1},\underline{k}_{n-2},i^{\underline{k}_{n-2}})$$

$$\tilde{d}_n(\underline{d}_{n-1},\underline{k}_{n-1},i^{\underline{k}_{n-1}}) \quad .$$

It should be emphasized that the decision rules must be retained to be used in any realization of the random decision process.

As a side product of the solution procedure, the maximum expectation of V - C is obtained. This expectation can be written

$$\max_{d_1}\Bigg[\max_{k_1}\Bigg[\int_{\Omega_{k_1}} f_{I^{k_1}|\underline{D}_1}(\,|\,)\max_{d_2}\Bigg[\cdots\max_{d_{n-1}}\Bigg[\max_{k_{n-1}}\Bigg[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)$$

$$\max_{d_n}\Bigg[\int_{\Omega_{\underline{X}_n}} V(\underline{x}_n,\underline{d}_n)\, f_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{n-1}}}(\,|\,)\, d\underline{x}_n\Bigg] di^{k_{n-1}} - C(k_{n-1})\Bigg]\Bigg] di^{k_{n-2}}$$

$$-C(k_{n-2})\Bigg]\Bigg]\ \cdots\ - C(k_1)\Bigg]\Bigg] = \max_{d_1}\Bigg[\int_{\Omega_{\underline{X}_1}} V_1(\underline{x}_1,\underline{d}_1)\, f_{\underline{X}_1|\underline{D}_1}(\,|\,)\, d\underline{x}_1$$

$$+\max_{k_1}\Bigg[\int_{\Omega_{k_1}} f_{I^{k_1}|\underline{D}_1}(\,|\,)\max_{d_2}\Bigg[\int_{\Omega_{\underline{X}_2}} V_2(\underline{x}_2,\underline{d}_2)\, f_{\underline{X}_2|\underline{D}_2,I^{k_1}}(\,|\,)\, d\underline{x}_2 + \cdots$$

$$+\max_{k_{n-2}}\Bigg[\int_{\Omega_{k_{n-2}}} f_{I^{k_{n-2}}|\underline{D}_{n-2},I^{\underline{k}_{n-3}}}(\,|\,)$$

$$\max_{d_{n-1}}\Bigg[\int_{\Omega_{\underline{X}_{n-1}}} V_{n-1}(\underline{x}_{n-1},\underline{d}_{n-1})\, f_{\underline{X}_{n-1}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)\, d\underline{x}_{n-1}$$

$$+\max_{k_{n-1}}\Bigg[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1},I^{\underline{k}_{n-2}}}(\,|\,)$$

$$\max_{d_n}\Bigg[\int_{\Omega_{\underline{X}_n}} V_n(\underline{x}_n,\underline{d}_n)\, f_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{n-1}}}(\,|\,)\, d\underline{x}_n\Bigg] di^{k_{n-1}}$$

$$-\,C(k_{n-1})\Bigg]\Bigg] di^{k_{n-2}} - C(k_{n-2})\Bigg]\ \cdots\ \Bigg]\Bigg] di^{k_1} - C(k_1)\Bigg]\Bigg] \quad .$$

Both of the above forms for the maximum expectation of V - C will be used in the remainder of this investigation. The first form is referred to as the condensed form. The second form is more natural to use in some applications and will be called the expanded form.

The sequence of events, as they occur, is indicated by the tree diagram shown in Figure 2.

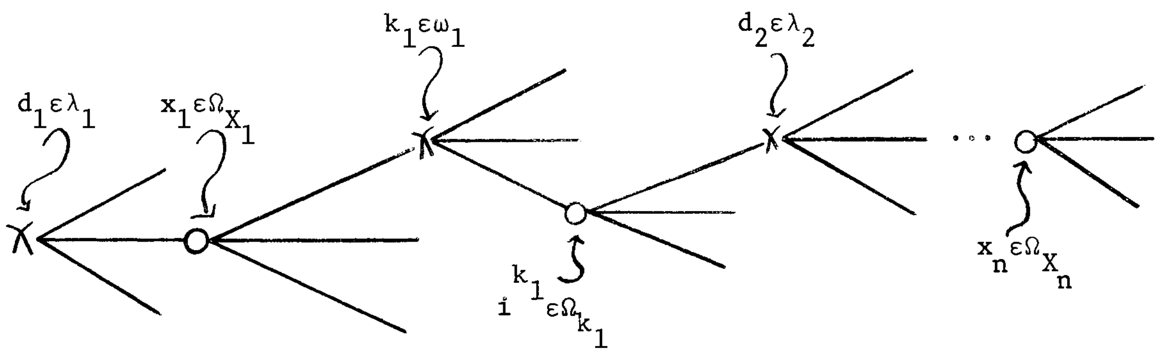

Figure 2. Tree Diagram of the General Formulation

It is not difficult to see that the solution of some seemingly simple problems fitting the formulation at the beginning of this chapter can be very complicated. For this reason, much of the work which follows is directed toward reducing solution computational effort.

Experiments Without Cost

In many problems the cost associated with experimentation is zero. Stated more precisely, for every $j = 1,2,\ldots,n-1$, $C(k_j) = 0$ for every $k_j \varepsilon \omega_j$. In this case, the following two questions are important in that they may lead to considerable simplification of the solution process.

(1) For each $j = 1,2,\ldots,n-1$ such that ω_j contains at least two elements, is it possible to eliminate from consideration the null experiment, $*j$?

(2) Is it possible to find a reasonably simple experiment characterization so that if for some $j = 1,2,\ldots,n-1$, there exists an experiment, $k_j' \varepsilon \omega_j$ with this characterization, then the optimal j^{th} experiment rule is k_j'?

Under a restriction on the density function,

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} \quad ,$$

the answer to both questions is yes. In regard to the second question, there is a characterization which is, in some ways, analogous to the statistic characterization known as a sufficient statistic (Hogg and Craig, 1966). The "yes" answers to the above questions are formalized with two corresponding theorems which provide in many problems, a great amount of computational simplification. The theorems are stated below and established in the Appendix. For examples of their application, the reader should refer to Chapter III and IV.

<u>Theorem 1</u>

Suppose that

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} = f_{I^{k_1}|\underline{X}_1} \, f_{I^{k_2}|\underline{X}_2} \cdots f_{I^{k_{n-1}}|\underline{X}_{n-1}} \quad .$$

Then for each $j = 1,2,\ldots,n-1$ such that $\omega_j \sim \{*j\}$ is not empty, denote

$\omega_j' = \omega_j \sim \{*j\}$. For each $j = 1,2,\ldots,n-1$ such that $\omega_j = \{*j\}$, denote $\omega_j' = \omega_j$.

The maximum expectation of V using $\omega_1',\omega_2',\ldots,\omega_{n-1}'$ as the sets of experiments is the same as the maximum expectation of V using $\omega_1,\omega_2,\ldots,\omega_{n-1}$.

<u>Theorem 2</u>

Suppose that

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} = f_{I^{k_1}|\underline{X}_1} f_{I^{k_2}|\underline{X}_2} \cdots f_{I^{k_{n-1}}|\underline{X}_{n-1}}$$

and that for some $j = 1,2,\ldots,n-1$, there exists a $k_j' \varepsilon \omega_j$ such that for every $k_j \varepsilon \omega_j$, $k_j \neq k_j'$, there exists a function, $z_{k_j}(i^{k_j},i^{k_j'})$ with the following properties:

(1) $z_{k_j}(i^{k_j},i^{k_j'}) \geq 0$

(2) $f_{I^{k_j}|\underline{X}_j}(|) = \int_{\Omega_{k_j'}} z_{k_j}(i^{k_j},i^{k_j'}) f_{I^{k_j'}|\underline{X}_j}(|)\, di^{k_j'}$

(3) $\int_{\Omega_{k_j}} z_{k_j}(i^{k_j},i^{k_j'})\, di^{k_j} = 1$

(4) $0 < \int_{\Omega_{k_j'}} z_{k_j}(i^{k_j},i^{k_j'})\, di^{k_j'} < \infty$.

The optimal j^{th} experiment rule is $\tilde{k}_j = k_j'$.

Notice that if the conditional density function of I^{k_j} given $I^{k_j'}$ and $\underline{X}_j$ can be obtained and that density function does not depend on $\underline{X}_j$ and satisfies property 4 of Theorem 2 then that density function may be used as the function z_{k_j} required in Theorem 2. The fact that the density must not depend on $\underline{X}_j$ suggests that the concept involved in Theorem 2 is analogous to the concept of a sufficient statistic. In fact, the concept involved in Theorem 2 has been referred to as a sufficient experiment (Blackwell, 1953; DeGroot, 1970).

Information Measure

A measure of the conditional information in an experiment, $k_{j-1}\varepsilon\omega_{j-1}$, will now be defined. Theorems 1 and 2 are then interpreted relative to the measure.

Let $U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})$ denote the maximum expectation of V given $\underline{d}_{j-1}$, $\underline{k}_{j-1}$, and $I^{\underline{k}_{j-1}} = i^{\underline{k}_{j-1}}$. A suitable definition of the conditional information, I, in an experiment $k_{j-1}\varepsilon\omega_{j-1}$ given $\underline{d}_{j-1}$, $\underline{k}_{j-2}$, and $I^{\underline{k}_{j-2}} = i^{\underline{k}_{j-2}}$ is given by the following equation,

$$
\begin{aligned}
&I(k_{j-1}|\underline{d}_{j-1}, i^{\underline{k}_{j-2}}) \\
&\quad \triangleq \int_{\Omega_{k_{j-1}}} f_{I^{k_{j-1}}|\underline{D}_{j-1}, I^{\underline{k}_{j-2}}}(|)\, U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})\, di^{k_{j-1}} \\
&\quad - \int_{\Omega_{*_{j-1}}} f_{I^{*_{j-1}}|\underline{D}_{j-1}, I^{\underline{k}_{j-2}}}(|)\, U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-2}}, I^{*_{j-1}}})\, di^{*_{j-1}} \\
&\quad = \int_{\Omega_{k_{j-1}}} f_{I^{k_{j-1}}|\underline{D}_{j-1}, I^{\underline{k}_{j-2}}}(|)\, U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})\, di^{k_{j-1}} - U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-2}}}) \\
&\quad = \int_{\Omega_{k_{j-1}}} f_{I^{k_{j-1}}|\underline{D}_{j-1}, I^{\underline{k}_{j-2}}}(|)\, [U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}}) - U_j(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-2}}})]\, di^{k_{j-1}}.
\end{aligned}
$$

From the definition it is seen that the conditional information in the null experiment, $*_{j-1}$, is zero. The conclusion of Theorem 1 is obtained because, under the hypothesis, the conditional information in an experiment, as defined above, is never negative. Reference to the proof of Theorem 1 will reveal that this is so. Thus, Theorem 1 may be interpreted as a statement that, under the hypothesis, the conditional information in an experiment is never negative.

If there is an experiment $k'_{j-1} \varepsilon \omega_{j-1}$ which satisfies Theorem 2, then regardless of what the state of knowledge $\underline{d}_{j-1}$, $\underline{k}_{j-2}$, and $i^{\underline{k}_{j-2}}$ may be, the conclusion of Theorem 2 may be interpreted as saying that k'_{j-1} has at least as much information as any other experiment in ω_{j-1}. It could be said that k'_{j-1} is a uniformly most informative experiment in ω_{j-1}.

Suppose in a particular problem that the hypothesis of Theorem 2 is satisfied and that for some $j = 1,2,\ldots,n-1$ the set of experiments ω_j is the set $\{\alpha_j,\beta_j,*_j\}$

$$f_{I^{\alpha_j}|\underline{X}_j}(w|\underline{x}_j) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{1}{2}\frac{(w-x_j)^2}{\sigma_1^2}}$$

$$f_{I^{\beta_j}|\underline{X}_j}(v|\underline{x}_j) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{1}{2}\frac{(v-x_j)^2}{\sigma_2^2}} \quad .$$

Then if $\sigma_2^2 > \sigma_1^2$,

$$z_{\beta_j}(v,w) = \frac{1}{\sqrt{2\pi(\sigma_2^2-\sigma_1^2)}} e^{-\frac{1}{2}\frac{(v-w)^2}{\sigma_2^2-\sigma_1^2}}$$

serves as the function called for in Theorem 2 that establishes that α_j is a uniformly most informative or sufficient experiment in ω_j.

Experiments With Cost

In the more general case when experiments are not without cost, methods which, in some cases, simplify the solution procedure can again be found although the simplifications are not usually as thorough as in the cost free case. In the work to follow, use will be made of a possibly hypothetical experiment, $\hat{k}_j$, which results in a sample of $I^{\hat{k}_j}$ where

$$I^{\hat{k}_j} = \underline{X}_j \qquad .$$

In other words, $\hat{k}_j$ is an experiment which may or may not actually be an element of ω_j and which results in exact knowledge as to the outcome of $\underline{X}_j$. Also, the notation

$$E^{k_p}_{max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

will be used to indicate the maximum expectation of

$$\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)$$

using the experiment k_p and given $\underline{d}_p$, $\underline{k}_{p-1}$, and $I^{\underline{k}_{p-1}} = i^{\underline{k}_{p-1}}$.

By reference to the expanded form of solution, it is easily seen that for an experiment, $k_p \varepsilon \omega_p$, to be a candidate for the optimal p^{th} experiment, the cost $C(k_p)$ must satisfy the inequality,

$$C(k_p) \leq E_{max}^{k_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

$$- E_{max}^{\overset{*}{p}}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\underline{d}_p, i^{\underline{k}_{p-1}}\right) \quad .$$

The right side of this inequality is a bound on the amount that could be spent for an experiment, $k_p \varepsilon \omega_p$, when the experiment yields information about $\underline{X}_p$ but the information has errors. Unfortunately, this bound provides little help in applications since obtaining the bound amounts to solving the problem by conventional means.

Two more theorems are now stated which give, in some cases, useful bounds on the maximum amount which can be spent on experimentation. The second theorem, Theorem 4, is a generalization of the concept discussed by Howard (1965) in relation to a particular systems engineering problem and reverifies the intuitive idea that the maximum amount which can be spent on experimentation can be no greater than the difference between the maximum expectation when an exact observation of $\underline{x}_j$ is available as a cost free experiment and the maximum expectation when the null experiment is used. It is shown in an example in Chapter V that the specific result obtained by Howard (1965) is a special case of the general idea expressed in Theorem 4.

Theorem 3

Suppose that

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} = f_{I^{k_1}|\underline{X}_1}\, f_{I^{k_2}|\underline{X}_2} \cdots f_{I^{k_{n-1}}|\underline{X}_{n-1}}$$

and that for some $p = 1,2,\ldots,n-1$, there exists a k_p', which may or may not be contained in ω_p, such that for every $k_p \varepsilon \omega_p$, $k_p \neq k_p'$ and $k_p \neq *_p$, there exists a function, $z_{k_p}(i^{k_p}, i^{k_p'})$, with the following properties:

(1) $z_{k_p}(i^{k_p}, i^{k_p'}) \geq 0$

(2) $f_{I^{k_p}|\underline{X}_p}(|) = \int_{\Omega_{k_p'}} z_{k_p}(i^{k_p}, i^{k_p'}) f_{I^{k_p'}|\underline{X}_p}(|)\, di^{k_p'}$

(3) $\int_{\Omega_{k_p}} z_{k_p}(i^{k_p}, i^{k_p'})\, di^{k_p} = 1$

(4) $0 < \int_{\Omega_{k_p'}} z_{k_p}(i^{k_p}, i^{k_p'})\, di^{k_p'} < \infty$.

Then the following inequality holds

$$C(\tilde{k}_p) \leq E^{k_p'}_{\max}\Big(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\Big)$$

$$- E^{*_p}_{\max}\Big(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\Big)$$

where $\tilde{k}_p$ is the optimal p^{th} experiment given $\underline{d}_p$, $\underline{k}_{p-1}$, and $I^{\underline{k}_{p-1}} = i^{\underline{k}_{p-1}}$.

If in a particular example each experiment $k_p \varepsilon \omega_p$, $k_p \neq *_p$, is of the form

$$I^{k_p} = \underline{X}_p + \underline{W}_{k_p} + \underline{V}_p$$

where $\underline{W}_{k_p}$ and $\underline{V}_p$ are random vectors, then the experiment k_p' defined by

$$I^{k_p'} = \underline{X}_p + \underline{V}_p$$

could be used in Theorem 3. In this example, $\underline{V}_p$ might represent an unavoidable measurement error associated with each experiment in ω_p.

<u>Theorem 4</u>

Suppose that

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} = f_{I^{k_1}|\underline{X}_1} \; f_{I^{k_2}|\underline{X}_2} \cdots f_{I^{k_{n-1}}|\underline{X}_{n-1}}$$

and for some p = 1,2,...,n-1, the inequality

$$0 < \int_{\Omega_{\underline{X}_p}} f_{I^{k_p}|\underline{X}_p}(|) \; d\underline{x}_p < \infty$$

holds for every $k_p \varepsilon \omega_p$, $k_p \neq *_p$.

Then the following inequality holds

$$C(\tilde{k}_p) \leq E_{\max}^{\hat{k}_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

$$- E_{\max}^{*_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

where $\tilde{k}_p$ is the optimal p^{th} experiment given $\underline{d}_p$, $\underline{k}_{p-1}$, and $I^{\underline{k}_{p-1}} = i^{\underline{k}_{p-1}}$.

Theorem 4 gives a bound on the maximum allowable expenditure for the p^{th} experiment given $\underline{d}_p$, $\underline{k}_{p-1}$, and $I^{\underline{k}_{p-1}} = i^{\underline{k}_{p-1}}$. The bound is obtained by computing the difference between the maximum expectation when an exact observation of $\underline{x}_p$ is obtained, free of cost, as a result of the p^{th} experiment and the maximum expectation when the null experiment is used as the p^{th} experiment. Since this bound represents the value of complete elimination of uncertainty as to the value of $\underline{X}_p$, it will be referred to as the conditional cost of uncertainty given $\underline{d}_p$, $\underline{k}_{p-1}$, and $I^{\underline{k}_{p-1}}$.

CHAPTER III

APPLICATION TO CONTROL

The purpose of this and the next two chapters is three fold. First, to emphasize the application of the problem formulation to practical problems. Second, to illustrate the general method of solution in some specific problems. Finally to demonstrate how Theorems 1 through 4 can be used to ease the computational burden in obtaining the solution to some specific problems. In this chapter the same basic control problem will be used in several examples to illustrate different points.

Example 1

Suppose there is a control system whose dynamics are specified by the first order linear difference equation

$$X_j = X_{j-1} + d_j, \qquad j = 1,2,\ldots,n \qquad .$$

X_j represents the state of the system at time j and d_j represents the decision at time j. The initial state of the system, X_0, is a normal random variable with zero mean and unit variance. For each j, $\lambda_j = R$, the set of real numbers.

For each j, the set of experiments, ω_j, consists of two elements, $*_j$ and α_j. Choice of the experiment α_j results in a sample of the random variable, I^{α_j}, where

$$I^{\alpha_j} = X_j + W_j \qquad .$$

W_j is a normal, zero mean, unit variance measurement noise. Also, W_j and W_k are independent if $j \neq k$ and X_0 and W_j are independent, $j = 1,2,\ldots,n-1$. Choice of the null experiment, $*_j$, results in a sample of the degenerate random variable, I^{*_j}, where

$$I^{*_j} = i^{*_j} \qquad .$$

It should be pointed out that in the problem formulation, the null experiment, $*_j$, could equivalently have been described as an experiment where nothing is observed. The general method of solution was, however, more easily formulated with the null experiment described as the observation of a degenerate random variable. In the present example, i^{*_j}, is the number that will be observed if the null experiment is chosen.

The experiment α_j has a cost of $C(\alpha_j) = \frac{1}{4}$, $j = 1,2,\ldots,n-1$. The utility functions are given to be $V_j(\underline{x}_j,\underline{d}_j) = -(x_j-y_j)^2$, $j = 1,2,\ldots,n$, where y_j is the desired state of the system at time j.

The decision and experiment rules and the maximum expectation of V - C will be found for n = 3.

The conditional expectation of V_3 given $\underline{d}_3$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$ can be written,

$$\int_{\Omega_{\underline{X}_3}} V_3(\underline{x}_3,\underline{d}_3) f(|)_{\underline{X}_3|\underline{D}_3, I^{\underline{k}_2}} \, d\underline{x}_3$$

$$= \int_{\Omega_{\underline{X}_3}} -(x_3-y_3)^2 f(|)_{\underline{X}_3|\underline{D}_3, I^{\underline{k}_2}} \, d\underline{x}_3 = \int_{\Omega_{X_2}} -(x_2+d_3-y_3)^2 f(|)_{X_2|\underline{D}_2, I^{\underline{k}_2}} \, dx_2 \ .$$

The required density is found by Bayes' rule to be

$$
f_{X_2|\underline{D}_2,I^{\underline{k}_2}}(x_2|\underline{d}_2,i^{\underline{k}_2}) = \begin{cases}
\dfrac{1}{\sqrt{2\pi\frac{1}{3}}}\, e^{-\dfrac{\left[x_2 - \frac{(d_1+d_2)+i^{\alpha_2}+(i^{\alpha_1}+d_2)}{3}\right]^2}{2(1/3)}}, & \text{if } \underline{k}_2 = (\alpha_1,\alpha_2) \\[2ex]
\dfrac{1}{\sqrt{2\pi\frac{1}{2}}}\, e^{-\dfrac{\left[x_2 - \frac{(d_1+d_2)+(i^{\alpha_1}+d_2)}{2}\right]^2}{2(1/2)}}, & \text{if } \underline{k}_2 = (\alpha_1,*_2) \\[2ex]
\dfrac{1}{\sqrt{2\pi\frac{1}{2}}}\, e^{-\dfrac{\left[x_2 - \frac{(d_1+d_2)+i^{\alpha_2}}{2}\right]^2}{2(1/2)}}, & \text{if } \underline{k}_2 = (*_1,\alpha_2) \\[2ex]
\dfrac{1}{\sqrt{2\pi}}\, e^{-\dfrac{[x_2 - (d_1+d_2)]^2}{2}}, & \text{if } \underline{k}_2 = (*_1,*_2).
\end{cases}
$$

The decision rule for d_3 is next obtained by computing the conditional expectation of V_3 given $\underline{d}_3$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$ and then differentiating and setting equal to zero. The result is:

$$\tilde{d}_3(\underline{d}_2,\underline{k}_2,i^{\underline{k}_2}) = \begin{cases} y_3 - \dfrac{(d_1+d_2) + i^{\alpha_2} + (i^{\alpha_1}+d_2)}{3}, & \text{if } \underline{k}_2 = (\alpha_1,\alpha_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + (i^{\alpha_1}+d_2)}{2}, & \text{if } \underline{k}_2 = (\alpha_1,*_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + i^{\alpha_2}}{2}, & \text{if } \underline{k}_2 = (*_1,\alpha_2) \\[2ex] y_3 - (d_1+d_2), & \text{if } \underline{k}_2 = (*_1,*_2). \end{cases}$$

Substitution of the rule $\tilde{d}_3$ into the conditional expectation of V_3 given $\underline{d}_3$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$ yields the maximum expectation of V_3 given $\underline{d}_2$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$,

$$\max_{d_3}\left[\int_{\Omega_{\underline{X}_3}} V_3(\underline{x}_3,\underline{d}_3) f(\,|\,)_{\underline{X}_3|\underline{D}_3,I^{\underline{k}_2}}\, d\underline{x}_3\right] = \begin{cases} -\frac{1}{3}, & \text{if } \underline{k}_2 = (\alpha_1,\alpha_2) \\[1ex] -\frac{1}{2}, & \text{if } \underline{k}_2 = (\alpha_1,*_2) \\[1ex] -\frac{1}{2}, & \text{if } \underline{k}_2 = (*_1,\alpha_2) \\[1ex] -1, & \text{if } \underline{k}_2 = (*_1,*_2) \end{cases} .$$

Notice that in this special case the maximum expectation of V_3 given $\underline{d}_2$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$ depends only upon the choice of $\underline{k}_2$.

Next, by simple comparison, the experiment rule for k_2 is found. The result is

$$\tilde{k}_2(\underline{d}_2,\underline{k}_1,i^{\underline{k}_1}) = \begin{cases} \alpha_2 \ , \text{ if } k_1 = *_1 \\ *_2 \ , \text{ if } k_1 = \alpha_1 \end{cases} .$$

Substitution of the rule $\tilde{k}_2$ into the conditional expectation of the maximum expectation of $V_3(\underline{x}_3,\underline{d}_3) - C(k_2)$ given $\underline{d}_2$, $\underline{k}_2$, $I^{\underline{k}_2} = i^{\underline{k}_2}$ given $\underline{d}_2$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$ yields the maximum expectation of $V_3(\underline{x}_3,\underline{d}_3) - C(k_2)$ given $\underline{d}_2$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$,

$$\max_{k_2}\left[\int_{\Omega_{k_2}} f_{I^{k_2}|\underline{D}_2,I^{k_1}}(\,|\,) \max_{d_3}\left[\int_{\Omega_{\underline{X}_3}} V_3(\underline{x}_3,\underline{d}_3) f_{\underline{X}_3|\underline{D}_3,I^{\underline{k}_2}}(\,|\,) d\underline{x}_3\right] di^{k_2} - C(k_2)\right]$$

$$= \begin{cases} -\frac{1}{2} \ , \text{ if } k_1 = \alpha_1 \\ -\frac{3}{4} \ , \text{ if } k_1 = *_1 \end{cases} .$$

The next step is to obtain the decision rule for d_2. Since the maximum expectation of $V_3(\underline{x}_3,\underline{d}_3) - C(k_2)$ given $\underline{d}_2$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$ does not depend on d_2, the decision rule for d_2 can be found by maximizing the conditional expectation of V_2 given $\underline{d}_2$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$. After using Bayes' rule to find the needed density function, the rule can be found by differentiating and setting equal to zero. The result is

$$\tilde{d}_2(\underline{d}_1,\underline{k}_1,i^{\underline{k}_1}) = \begin{cases} y_2 - \dfrac{d_1 + i^{\alpha_1}}{2} & , \text{ if } \underline{k}_1 = \alpha_1 \\ y_2 - d_1 & , \text{ if } \underline{k}_1 = *_1 \end{cases} \quad .$$

Substitution of the rule $\tilde{d}_2$ yields the maximum expectation of $V_2(\underline{x}_2,\underline{d}_2) + V_3(\underline{x}_3,\underline{d}_3) - C(k_2)$ given $\underline{d}_1$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$,

$$\max_{d_2}\left[\max_{k_2}\left[\int_{\Omega_{k_2}} f_{I^{k_2}|\underline{D}_2,I^{k_1}}(\,|\,)\max_{d_3}\left[\int_{\Omega_{\underline{X}_3}}\left[V_2(\underline{x}_2,\underline{d}_2)+V_3(\underline{x}_3,\underline{d}_3)\right] f_{\underline{X}_3|\underline{D}_3,I^{\underline{k}_2}}(\,|\,)\,d\underline{x}_3\right] di^{k_2}\right.\right.$$

$$\left.\left. + C(k_2)\right]\right] = \begin{cases} -1 & , \text{ if } \underline{k}_1 = \alpha_1 \\ -1\frac{3}{4} & , \text{ if } \underline{k}_1 = *_1 \end{cases} \quad .$$

Again, by simple comparison, the experiment rule for k_1 is found. The result is

$$\tilde{k}_1(d_1) = \alpha_1 \quad .$$

The maximum expectation of $V_2(\underline{x}_2,\underline{d}_2)+V_3(\underline{x}_3,\underline{d}_3)-C(k_2)-C(k_3)$ given $\underline{d}_1$ is found by substitution of the rule $\tilde{k}_1$ to be -1. Since this maximum expectation does not involve d_1, the decision rule for d_1 can be found by maximizing the expectation of $V_1(\underline{x}_1,\underline{d}_1)$ given $\underline{d}_1$. The decision rule is

$$\tilde{d}_1 = y_1 \quad .$$

The maximum expectation of V - C is finally found to be -2.

A summary of the decision and experiment rules is given below:

$$\tilde{d}_1 = y_1$$

$$\tilde{k}_1 = \alpha_1$$

$$\tilde{d}_2 = y_2 - \frac{y_1 + i^{\alpha_1}}{2}$$

$$\tilde{k}_2 = *_2$$

$$\tilde{d}_3 = y_3 - y_2 \quad .$$

The fact that d_3 is determined before the decision process starts is due to the fact that no new information is obtained after the experiment k_1 is performed, no new uncertainty is introduced after the experiment k_1 is performed, and d_2 is chosen so as to use the information from experiment k_1.

Example 2

In this example, the decision and experiment rules and the maximum expectation of V - C will be found for the control problem of Example 1 with the change, $C(\alpha_j) = 0$, $j = 1,2,\ldots,n-1$. The solution will be obtained for arbitrary n.

By Theorem 1, the optimal experiment rules are $\tilde{k}_1=\alpha_1, \tilde{k}_2=\alpha_2, \ldots,$ $\tilde{k}_{n-1}=\alpha_{n-1}$.

The expectation of V_n given $\underline{d}_n$, $\underline{k}_{n-1}$, and $I^{\underline{k}_{n-1}} = i^{\underline{k}_{n-1}}$ can be written

$$\int_{\Omega_{\underline{X}_n}} V_n(\underline{x}_n,\underline{d}_n)\, f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|)\, d\underline{x}_n = \int_{\Omega_{\underline{X}_n}} -(x_n-y_n)^2 f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|)\, d\underline{x}_n$$

$$= - \int_{\Omega_{X_{n-1}}} (x_{n-1}+d_n-y_n)^2 f_{X_{n-1}|\underline{D}_{n-1}, I^{\underline{k}_{n-1}}}(|)\, dx_{n-1} \quad .$$

With $\underline{k}_{n-1} = \tilde{\underline{k}}_{n-1} = \underline{\alpha}_{n-1}$, the required density function is found using Bayes' rule.

$$f_{X_{n-1}|\underline{D}_{n-1}, I^{\underline{\alpha}_{n-1}}}(|)$$

$$= \frac{1}{\sqrt{2\pi\frac{1}{n}}}\, e^{-\frac{\left[x_{n-1} - \frac{\sum_{j=1}^{n-1} d_j + i^{\alpha_{n-1}} + (i^{\alpha_{n-2}} + d_{n-1}) + \ldots + (i^{\alpha_1} + \sum_{j=2}^{n-1} d_j)}{n}\right]^2}{2(1/n)}} \quad .$$

The decision rule for d_n is found by computing the conditional expectation of V_n given $\underline{d}_n$, $\underline{\alpha}_{n-1}$, and $I^{\underline{\alpha}_{n-1}} = i^{\underline{\alpha}_{n-1}}$, differentiating with respect to d_n and setting equal to zero. The result is

$$\check{d}_n(\underline{d}_{n-1}, \underline{\alpha}_{n-1}, i^{\underline{\alpha}_{n-1}})$$

$$= y_n - \frac{\sum_{j=1}^{n-1} d_j + i^{\alpha_{n-1}} + (i^{\alpha_{n-2}} + d_{n-1}) + \ldots + (i^{\alpha_1} + \sum_{j=2}^{n-1} d_j)}{n} \quad .$$

Substitution of the rule $\check{d}_n$ into the expectation of V_n given $\underline{d}_n$, $\underline{\alpha}_{n-1}$, and $I^{\underline{\alpha}_{n-1}} = i^{\underline{\alpha}_{n-1}}$ yields the maximum expectation of V_n given $\underline{d}_{n-1}$, $\underline{\alpha}_{n-1}$, and $I^{\underline{\alpha}_{n-1}} = i^{\underline{\alpha}_{n-1}}$,

$$\max_{d_n}\left[-\int_{\Omega_{X_{n-1}}} (x_{n-1} + d_n - y_n)^2 \, f(\,|\,)_{X_{n-1}|\underline{D}_{n-1}, I^{\underline{\alpha}_{n-1}}} \, dx_{n-1}\right] = -\frac{1}{n} \quad .$$

The conditional expectation of the maximum expectation of V_n given $\underline{d}_{n-1}$, $\underline{\alpha}_{n-1}$, and $I^{\underline{\alpha}_{n-1}} = i^{\underline{\alpha}_{n-1}}$ given $\underline{d}_{n-1}$, $\underline{\alpha}_{n-2}$, and $I^{\underline{\alpha}_{n-2}} = i^{\underline{\alpha}_{n-2}}$ is also $-1/n$. This result is not a function of d_{n-1} and the decision rule for d_{n-1} is determined by maximizing the expectation of V_{n-1} given $\underline{d}_{n-1}$, $\underline{\alpha}_{n-2}$, and $I^{\underline{\alpha}_{n-2}} = i^{\underline{\alpha}_{n-2}}$ with respect to d_{n-1}. Continuing this line of reasoning, it is easy to conclude that for any $p = 1,2,\ldots,n$, the decision rule for d_p will depend only upon maximizing the expectation of V_p given $\underline{d}_p$, $\underline{\alpha}_{p-1}$, and $I^{\underline{\alpha}_{p-1}} = i^{\underline{\alpha}_{p-1}}$ with respect to d_p.

Using Bayes' rule to find the pertinent density function, the general expression for $\overset{\gamma}{d}_p$, $p = 1,2,\ldots,n$, can be found.

$$\overset{\gamma}{d}_p = y_1 \ , \text{ if } p = 1$$

$$\overset{\gamma}{d}_p(\underline{d}_{p-1}, \underline{\alpha}_{p-1}, i^{\underline{\alpha}_{p-1}})$$

$$= y_p - \frac{\sum_{j=1}^{p-1} d_j + i^{\alpha_{p-1}} + (i^{\alpha_{p-2}} + d_{p-1}) + \ldots + (i^{\alpha_1} + \sum_{j=2}^{p-1} d_j)}{p} \ ,$$

$$\text{if } p = 2,3,\ldots,n.$$

The maximum expectation of V using the optimum decision and experiment rules is $-\frac{1}{n} - \frac{1}{n-1} - \frac{1}{n-2} - \ldots - \frac{1}{2} - 1$. If the null experiment, $*_j$, were used for each $j = 1,2,\ldots,n-1$, the maximum expectation of V would be $-n$. The improvement in expectation of V is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n}$.

Recall from Example 1 that when $n = 3$ and $C(\alpha_j) = \frac{1}{4}$, the maximum expectation of V - C was -2. In the present example, when n = 3 the maximum expectation of V = V - C is $-\frac{11}{6}$ which is greater than -2. However, were the two experiments α_1 and α_2 to cost $\frac{1}{4}$ each, the expectation of $V - \frac{1}{2}$ would be $-2\frac{1}{3}$ which is less than -2.

Example 3

In this example, the following changes are to be made in the problem formulation of Example 1.

The initial state of the system, X_0, has the density function,

$$f_{X_0}(x_0) = \begin{cases} e^{-x_0} & , x_0 \geq 0 \\ \\ 0 & , \text{otherwise} \end{cases} .$$

For each j = 1,2,...,n-1, the set of experiments, ω_j, contains three elements, $*_j$, k'_j, and k''_j.

$$I^{k'_j} = \min(0_{j1}, 0_{j2})$$

$$I^{k''_j} = \max(0_{j1}, 0_{j2})$$

where

$$0_{j1} = X_j + W_{j1}$$

$$0_{j2} = X_j + W_{j2} .$$

W_{j1} and W_{j2} are independent and have the density function

$$f_{W_{j1}}(w) = f_{W_{j2}}(w) = \begin{cases} e^{-w} & , w \geq 0 \\ 0 & , \text{otherwise} \end{cases}$$

$$C(k_j) = 0 \ \forall \ k_j \varepsilon \omega_j \text{ and } \forall \ j = 1,2,\ldots,n-1 \qquad .$$

Since W_{j1} and W_{j2} are independent, the joint density function of 0_{j1} and 0_{j2} given $X_j = x_j$ is

$$f_{0_{j1},0_{j2}|X_j}(w,v|x_j) = \begin{cases} e^{-(w-x_j)} e^{-(v-x_j)} & , w,v \geq x_j \\ 0 & , \text{otherwise} \end{cases} \qquad .$$

Using the theory of order statistics, it is possible to find the conditional density of $I^{k''_j}$ given $I^{k'_j} = i^{k'_j}$ and $X_j = x_j$. The result is

$$f_{I^{k''_j}|I^{k'_j},X_j}(i^{k''_j}|i^{k'_j},x_j) = \begin{cases} e^{-(i^{k''_j}-i^{k'_j})} & , i^{k'_j} < i^{k''_j} < \infty \\ 0 & , \text{otherwise} \end{cases} \qquad .$$

By Theorem 1, $*_j$ may be discarded from consideration for each j. The conditional density of $I^{k''_j}$ given $I^{k'_j} = i^{k'_j}$ and $X_j = x_j$ may be used as the function $z_{k''_j}$ in Theorem 2 to establish that $\underline{\tilde{k}}_{n-1} = \underline{k}'_{n-1}$.

Given a numerical value for n, the optimal decision rules for this example can be found using the same procedure as in Examples 1 and 2. For n = 2, the pertinent details of the solution are given below:

$$\max_{d_2} \int_{\Omega_{\underline{X}_2}} V_2(\underline{x}_2,\underline{d}_2)\, f_{\underline{X}_2|\underline{D}_2,I^{\underline{k}_1'}}(|)\, d\underline{x}_2 = \frac{(d_1 - i^{k_1'})^2\, e^{(i^{k_1'} + d_1)}}{(e^{i^{k_1'}} - e^{d_1})^2} - 1$$

$$\tilde{d}_2(\underline{d}_1,\underline{k}_1',i^{\underline{k}_1'}) = y_2 - \frac{(i^{k_1'} - 1)\, e^{i^{k_1'}} - (d_1 - 1)\, e^{d_1}}{e^{i^{k_1'}} - e^{d_1}}$$

$$\tilde{k}_1(\underline{d}_1) = k_1'$$

$$\max_{d_1} \int_{\Omega_{\underline{X}_1}} V_1(\underline{x}_1,\underline{d}_1)\, f_{\underline{X}_1|\underline{D}_1}(|)\, d\underline{x}_1 = -1$$

$$\tilde{d}_1 = y_1 - 1 \quad .$$

Example 4

In this example, the control problem of Example 1 is again considered, but with the following changes.

The initial state of the system, X_0, has the density function,

$$f_{X_0}(x_0) = \begin{cases} 1 , & -\frac{1}{2} \le x_0 \le \frac{1}{2} \\ 0 , & \text{otherwise} \end{cases} \quad .$$

For each $j = 1,2,\ldots,n-1$, there are three observations available:

$$O_{j1} = X_j + W_{j1}$$

$$O_{j2} = X_j + W_{j2}$$

$$0_{j3} = X_j + W_{j3}$$

where W_{j1}, W_{j2}, and W_{j3} are independent random variables having the density function

$$f_{W_{j1}}(w) = f_{W_{j2}}(w) = f_{W_{j3}}(w) = \begin{cases} 1 \ , \ -\frac{1}{2} \leq w \leq \frac{1}{2} \\ \\ 0 \ , \ \text{otherwise} \end{cases} .$$

Existing hardware dictates that for each j, the set of experiments, ω_j, contains the three elements, $*_j$, k_j', and k_j'', where

$$I^{k_j'} = (I_1^{k_j'}, I_2^{k_j'}) = (\min(0_{j1}, 0_{j2}, 0_{j3}), \max(0_{j1}, 0_{j2}, 0_{j3}))$$

and

$$I^{k_j''} = \frac{0_{j1} + 0_{j2} + 0_{j3}}{3} .$$

The cost associated with k_j' and k_j'' is zero for every $j = 1,2,\ldots,n-1$.

Let

$$I_3^{k_j'} = \text{median}(0_{j1}, 0_{j2}, 0_{j3}) .$$

Using the theory of order statistics, the joint density of $I_1^{k_j'}$, $I_2^{k_j'}$, and $I_3^{k_j'}$ given $X_j = x_j$ can be found. The result is

$$f_{I_1^{k_j'}, I_2^{k_j'}, I_3^{k_j'} | X_j}(w,x,y|x_j) = \begin{cases} 3! \ , \ x_j - \frac{1}{2} \leq w \leq y \leq x \leq x_j + \frac{1}{2} \\ \\ 0 \ \ , \ \text{otherwise} \end{cases} .$$

From this result, the density of $I_3^{k_j'}$ given $I_1^{k_j'} = w$, $I_2^{k_j'} = x$, and $X_j = x_j$ is found to be

$$f_{I_3^{k_j'} | I_1^{k_j'}, I_2^{k_j'}, X_j}(y|w,x,x_j) = \begin{cases} \dfrac{1}{x - w}, & w \le y \le x \\ 0, & \text{otherwise} \end{cases} .$$

The next goal is to find the density of $I^{k_j''}$ given $I^{k_j'} = i^{k_j'}$ and $X_j = x_j$.

$$\text{Prob}(I^{k_j''} \le i^{k_j''} | I_1^{k_j'} = w,\ I_2^{k_j'} = x,\ X_j = x_j) = \int_w^{3i^{k_j''} - w - x} \frac{1}{x-w}\, dy .$$

By differentiating with respect to $i^{k_j''}$, the desired density function is obtained.

$$f_{I^{k_j''} | I_1^{k_j'}, I_2^{k_j'}, X_j}(i^{k_j''} | i_1^{k_j'}, i_2^{k_j'}, x_j)$$

$$= \begin{cases} \dfrac{3}{i_2^{k_j'} - i_1^{k_j'}}, & \frac{2}{3} i_1^{k_j'} + \frac{1}{3} i_2^{k_j'} \le i^{k_j''} \le \frac{1}{3} i_1^{k_j'} + \frac{2}{3} i_2^{k_j'} \\ 0, & \text{otherwise} \end{cases} .$$

By Theorem 1, $*_j$ may be discarded from consideration for each j. The conditional density of $I^{k_j''}$ given $I^{k_j'} = i^{k_j'}$ and $X_j = x_j$ may be used as the function $z_{k_j''}$ in Theorem 2 to establish that $\underset{\sim}{\tilde{k}}_{n-1} = \underset{\sim}{k'}_{n-1}$.

Example 5

The control problem first introduced in Example 1 will be used one last time to illustrate the application of Theorem 3. Consider the same problem stated in Example 1 except as noted below.

For each $j = 1,2,\ldots,n-1$, the set of experiments, ω_j, will consist of the elements $*_j, \sigma_{1,j}, \sigma_{2,j}, \sigma_{3,j}, \ldots$. As usual, $*_j$ denotes the null experiment. The random variable observed as a result of performing the experiment $\sigma_{p,j}$, $p = 1,2,3,\ldots$, $j = 1,2,\ldots,n-1$, can be described by the equation

$$I^{\sigma_{p,j}} = X_j + W_j + Z_{p,j}$$

where W_j is a normal, zero mean, unit variance random variable for each j and $Z_{p,j}$ is a normal, zero mean, random variable with variance $\sigma^2_{p,j}$. X_0, W_j, and $Z_{p,j}$ are independent for each j and p. Also, W_j and W_k are independent for $j \neq k$ and $Z_{p,j}$ and $Z_{q,k}$ are independent for $j \neq k$ and each p and q. Thus, for a given j, the experiments $\sigma_{p,j}$, $p = 1,2,\ldots$, have two sources of error, W_j and $Z_{p,j}$. There is no control over the source of error, W_j, but by the choice of experiment the source of error, $Z_{p,j}$, can be controlled. A physical interpretation of the situation might be that W_j represents an inherent measurement error and $Z_{p,j}$ represents a controllable experimental error.

In the discussion which follows, it will be assumed that if $p > q$, $\sigma^2_{p,j} \leq \sigma^2_{q,j}$. Also, if $\sigma^2_{p,j} \geq \sigma^2_{q,j}$, then $C(\sigma_{p,j}) \leq C(\sigma_{q,j})$. Thus, if $p > q$, then $C(\sigma_{p,j}) \geq C(\sigma_{q,j})$. C is unbounded for every j. That is, for any given j and any real number M, there exists an interger p such that $C(\sigma_{p,j}) > M$. Also, for any positive number N and any $j = 1,2,\ldots$, $n-1$ there esists an integer q such that $\sigma^2_{q,j} < N$. These statements imply that by choice of experiment the variance of $Z_{p,j}$ can be made as small as desired, but in doing so the price will become arbitrarily large.

Since there are an infinite number of experiments in each ω_j, direct solution of the problem by backward induction would be very difficult. Theorem 3 can be used, however, to eliminate from consideration all but a finite number of experiments from each ω_j. The point of departure for solving the problem will then be an equivalent problem with only a finite number of experiments to consider for each $j = 1,2,\ldots,$ $n-1$.

From Theorem 3 a bound for the maximum amount which could be spent on the $n-1^{st}$ experiment given $\underline{d}_{n-1}$, $\underline{k}_{n-2}$, and $I^{\underline{k}_{n-2}} = i^{\underline{k}_{n-2}}$ can be written

$$C(\tilde{k}_{n-1}) \le E_{\max}^{\hat{k}_{n-1}}(V_n(\underline{x}_n,\underline{d}_n)\,|\,\underline{d}_{n-1}, i^{\underline{k}_{n-2}}) - E_{\max}^{*\,n-1}(V_n(\underline{x}_n,\underline{d}_n)\,|\,\underline{d}_{n-1}, i^{\underline{k}_{n-2}}) \quad .$$

Since $V_n(\underline{x}_n,\underline{d}_n) = -(x_n - y_n)^2 = -(x_{n-1}+d_n-y_n)^2$, an exact observation for the $n-1^{st}$ experiment will yield, with the correct decision, zero error between x_n and y_n or $V_n(\underline{x}_n,\underline{d}_n) = 0$. Therefore,

$$E_{\max}^{\hat{k}_{n-1}}(V_n(\underline{x}_n,\underline{d}_n)\,|\,\underline{d}_{n-1}, i^{\underline{k}_{n-2}}) = 0$$

for any $\underline{d}_{n-1}$, $\underline{k}_{n-2}$, and $i^{\underline{k}_{n-2}}$.

Since the initial state and experiment errors are independent and normal, the probability density function for X_{n-1} given $\underline{d}_{n-1}$, $\underline{k}_{n-2}$, and $I^{\underline{k}_{n-2}} = i^{\underline{k}_{n-2}}$ will be normal with variance less than or equal to one. $-E_{\max}^{*\,n-1}(V_n(\underline{x}_n,\underline{d}_n)\,|\,\underline{d}_{n-1}, i^{\underline{k}_{n-2}})$ is equal to the variance of X_{n-1} given

$\underset{-}{d}_{n-1}$, $\underset{-}{k}_{n-2}$, and $I^{\underset{-}{k}_{n-2}} = i^{\underset{-}{k}_{n-2}}$. Thus, the following inequality is obtained,

$$C(\tilde{k}_{n-1}) \leq 0 + 1 = 1 \qquad .$$

Since for some P, $p > P$ implies that $C(\sigma_{p,n-1}) > 1$, the set of experiments, ω_{n-1}, can be replaced by the set of experiments, $\omega'_{n-1} = \{*_{n-1}, \sigma_{1,n-1}, \sigma_{2,n-1}, \ldots, \sigma_{P,n-1}\}$.

Using Theorem 3 once again, a bound for the maximum amount which could be spent on the n-2nd experiment given $\underset{-}{d}_{n-2}$, $\underset{-}{k}_{n-3}$, and $I^{\underset{-}{k}_{n-3}} = i^{\underset{-}{k}_{n-3}}$ can be written

$$C(\tilde{k}_{n-2}) \leq E_{max}^{\hat{k}_{n-2}}(V_n(\underset{-}{x}_n,\underset{-}{d}_n) + V_{n-1}(\underset{-}{x}_{n-1},\underset{-}{d}_{n-1}) - C(k_{n-1})\,|\,\underset{-}{d}_{n-2}, i^{\underset{-}{k}_{n-3}})$$

$$- E_{max}^{*_{n-2}}(V_n(\underset{-}{x}_n,\underset{-}{d}_n) + V_{n-1}(\underset{-}{x}_{n-1},\underset{-}{d}_{n-1}) - C(k_{n-1})\,|\,\underset{-}{d}_{n-2}, i^{\underset{-}{k}_{n-3}}) \qquad .$$

An exact observation for the n-2nd experiment will yield, with the correct decisions, zero error between x_n and y_n and between x_{n-1} and y_{n-1} with zero expenditure for the k-1st experiment. Therefore,

$$E_{max}^{\hat{k}_{n-2}}(V_n(\underset{-}{x}_n,\underset{-}{d}_n) + V_{n-1}(\underset{-}{x}_{n-1},\underset{-}{d}_{n-1}) - C(k_{n-1})\,|\,\underset{-}{d}_{n-2}, i^{\underset{-}{k}_{n-3}}) = 0 \qquad .$$

Since the initial state and experiment errors are independent and normal, the probability density function for X_{n-1} given $\underset{-}{d}_{n-1}$, $\underset{-}{k}_{n-3}$, k_{n-1}, $I^{\underset{-}{k}_{n-3}} = i^{\underset{-}{k}_{n-3}}$, and $I^{\underset{-}{k}_{n-1}} = i^{\underset{-}{k}_{n-1}}$ will be normal with variance less than or equal to one. Similarly, the probability density function for X_{n-2} given $\underset{-}{d}_{n-2}$, $\underset{-}{k}_{n-3}$, and $I^{\underset{-}{k}_{n-3}} = i^{\underset{-}{k}_{n-3}}$ will be normal with variance less than or equal to one. Since $C(\tilde{k}_{n-1}) \leq 1$, the following inequality is obtained,

$$C(\tilde{k}_{n-2}) \leq 3 \quad .$$

Since for some Q, $p > Q$ implies that $C(\sigma_{p,n-2}) \geq 3$, the set of experiments, ω_{n-2}, can be replaced by the set of experiments, $\omega'_{n-2} = \{*_{n-2}, \sigma_{1,n-2}, \sigma_{2,n-2}, \ldots, \sigma_{Q,n-2}\}$.

Continuing in the above fashion, the following bounds can be obtained,

$$C(\tilde{k}_{n-3}) \leq 6$$

$$C(\tilde{k}_{n-4}) \leq 10$$

$$\vdots$$

$$C(\tilde{k}_p) \leq n-p + \sum_{j=1}^{n-p-1} j = n-p + \frac{(n-p-1)(n-p)}{2}$$

$$\vdots$$

$$C(\tilde{k}_1) \leq n-1 + \sum_{j=1}^{n-2} j = n-1 + \frac{(n-2)(n-1)}{2} \quad .$$

These bounds on the maximum amount which could be spent on experimentation determine the corresponding finite sets of experiments, $\omega'_{n-3}, \omega'_{n-4}, \ldots, \omega'_1$. Solution of this equivalent problem may still be formidable, but the procedure is now clear.

Suppose now that for each j, $\sigma^2_{p,j} = \frac{1}{p}$ and $C(\sigma_{p,j}) = p$. Suppose also that $n = 3$. Using the results obtained above, $C(\tilde{k}_2) \leq 1$, $C(\tilde{k}_1) \leq 3$, $\omega'_2 = \{*_2, \sigma_{1,2}\}$, $\omega'_1 = \{*_1, \sigma_{1,1}, \sigma_{2,1}, \sigma_{3,1}\}$.

Using Bayes' rule and the usual procedure of differentiating and setting equal to zero, the decision rule for d_3 is obtained.

$$\tilde{d}_3 = \begin{cases} y_3 - (d_1+d_2) & , \quad \text{if } \underline{k}_2 = (*_1, *_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{1}{2}\,(i^{\sigma_{1,1}}+d_2)}{1 + 1/2} & , \quad \text{if } \underline{k}_2 = (\sigma_{1,1}, *_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{2}{3}\,(i^{\sigma_{2,1}}+d_2)}{1 + 2/3} & , \quad \text{if } \underline{k}_2 = (\sigma_{2,1} *_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{3}{4}\,(i^{\sigma_{3,1}}+d_2)}{1 + 3/4} & , \quad \text{if } \underline{k}_2 = (\sigma_{3,1}, *_2) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{1}{2}\, i^{\sigma_{1,2}}}{1 + 1/2} & , \quad \text{if } \underline{k}_2 = (*_1, \sigma_{1,2}) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{1}{2}\, i^{\sigma_{1,2}} + \frac{1}{2}\,(i^{\sigma_{1,1}}+d_2)}{1 + 1/2 + 1/2} & , \quad \text{if } \underline{k}_2 = (\sigma_{1,1}, \sigma_{1,2}) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{1}{2}\, i^{\sigma_{1,2}} + \frac{2}{3}\,(i^{\sigma_{2,1}}+d_2)}{1 + 1/2 + 2/3} & , \quad \text{if } \underline{k}_2 = (\sigma_{2,1}, \sigma_{1,2}) \\[2ex] y_3 - \dfrac{(d_1+d_2) + \frac{1}{2}\, i^{\sigma_{1,2}} + \frac{3}{4}\,(i^{\sigma_{3,1}}+d_2)}{1 + 1/2 + 3/4} & , \quad \text{if } \underline{k}_2 = (\sigma_{3,1}, \sigma_{1,2}). \end{cases}$$

Next, the maximum expectation of V_3 given $\underline{d}_2$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$ is found by substitution of the rule $\tilde{d}_3$ into the expectation of V_3 given $\underline{d}_3$, $\underline{k}_2$, and $I^{\underline{k}_2} = i^{\underline{k}_2}$. The result is

$$\max_{\underline{d}_3}\left[\int_{\Omega_{\underline{X}_3}} V_3(\underline{x}_3,\underline{d}_3)\, f_{\underline{X}_3|\underline{D}_3,I}(\,|\,)\; \underline{k}_2\, d\underline{x}_3\right] = \begin{cases} -1 & \text{, if } \underline{k}_2 = ({*}_1,{*}_2) \\ -\frac{2}{3} & \text{, if } \underline{k}_2 = (\sigma_{1,1},{*}_2) \\ -\frac{3}{5} & \text{, if } \underline{k}_2 = (\sigma_{2,1},{*}_2) \\ -\frac{4}{7} & \text{, if } \underline{k}_2 = (\sigma_{3,1},{*}_2) \\ -\frac{2}{3} & \text{, if } \underline{k}_2 = ({*}_1,\sigma_{1,2}) \\ -\frac{1}{2} & \text{, if } \underline{k}_2 = (\sigma_{1,1},\sigma_{1,2}) \\ -\frac{6}{13} & \text{, if } \underline{k}_2 = (\sigma_{2,1},\sigma_{1,2}) \\ -\frac{4}{9} & \text{, if } \underline{k}_2 = (\sigma_{3,1},\sigma_{1,2}) \end{cases} .$$

By simple comparison, the experiment rule for k_2 is found.

$$\tilde{k}_2 = {*}_2 \qquad .$$

The maximum expectation of $V_3(\underline{x}_3,\underline{d}_3)-C(k_2)$ is found by the usual procedure.

$$\max_{k_2}\left[\int_{\Omega_{k_2}} f_{I^{k_2}|\underline{D}_2, I^{\underline{k}_1}}(|) \max_{d_3}\left[\int_{\Omega_{\underline{X}_3}} V_3(\underline{x}_3,\underline{d}_3) f_{\underline{X}_3|\underline{D}_3, I^{\underline{k}_2}}(|)\, d\underline{x}_3\right] - C(k_2)\right]$$

$$= \begin{cases} -1, & \text{if } k_1 = *_1 \\ -\frac{2}{3}, & \text{if } k_1 = \sigma_{1,1} \\ -\frac{3}{5}, & \text{if } k_1 = \sigma_{2,1} \\ -\frac{4}{7}, & \text{if } k_1 = \sigma_{3,1} \end{cases} .$$

The decision rule for d_2 is also found by the usual procedure.

$$\tilde{d}_2 = \begin{cases} y_2 - d_1 & , \text{ if } k_1 = *_1 \\ y_2 - \dfrac{d_1 + \frac{1}{2}\, i^{\sigma_{1,1}}}{1 + 1/2} & , \text{ if } k_1 = \sigma_{1,1} \\ y_2 - \dfrac{d_1 + \frac{2}{3}\, i^{\sigma_{2,1}}}{1 + 2/3} & , \text{ if } k_1 = \sigma_{2,1} \\ y_2 - \dfrac{d_1 + \frac{3}{4}\, i^{\sigma_{3,1}}}{1 + 3/4} & , \text{ if } k_1 = \sigma_{3,1} \end{cases} .$$

Use of the rule $\tilde{d}_2$ yields the maximum expectation of $V_2(\underline{x}_2,\underline{d}_2)+V_3(\underline{x}_3,\underline{d}_3)-C(k_2)$ given $\underline{d}_1$, $\underline{k}_1$, and $I^{\underline{k}_1} = i^{\underline{k}_1}$.

$$\max_{d_2}\left[\max_{k_2}\left[\int_{\Omega_{k_2}} f_{I^{k_2}|\underline{D}_2, I^{\underline{k}_1}}(\,|\,)\right.\right.$$

$$\left.\left.\max_{d_3}\left[\int_{\Omega_{\underline{X}_3}} [V_2(\underline{x}_2,\underline{d}_2)+V_3(\underline{x}_3,\underline{d}_3)-C(k_2)]\,f_{\underline{X}_3|\underline{D}_3, I^{\underline{k}_2}}(\,|\,)\,d\underline{x}_3\right] di^{k_2}\right]\right]$$

$$= \begin{cases} -2\ , & \text{if } k_1 = *_1 \\ -\frac{4}{3}\ , & \text{if } k_1 = \sigma_{1,1} \\ -\frac{6}{5}\ , & \text{if } k_1 = \sigma_{2,1} \\ -\frac{8}{7}\ , & \text{if } k_1 = \sigma_{3,1} \end{cases} \qquad .$$

Thus, the experiment rule for k_1 is $\tilde{k}_1 = *_1$. The maximum expectation of $V_2 + V_3 - C$ given $\underline{d}_1$ is -2. The decision rule for d_1 is easily found to be

$$\tilde{d}_1 = y_1 \qquad .$$

The maximum expectation of $V - C$ is -3.

CHAPTER IV

APPLICATION TO WEAPON ANALYSIS

With the exception of occasional reference to the theorems of Chapter II, this chapter is written so as to be self contained. This is done so that a reader who is interested only in the weapon analysis application of the general ideas which have been developed can read the present chapter independently of others. For the most part, the material in this chapter is a direct application of the ideas developed in Chapter II; however, there is a small amount of new material which is encountered for the first time in this chapter and has application only to the particular problem being considered. Since part of the work in this chapter is based on the theoretical approach to target coverage problems developed by Snow (1966), an attempt has been made to use, in as much of the work as possible, notation identical with that used by Snow. The set of notation used in this chapter is therefore inconsistent with the set of notation used in previous chapters; however, an attempt will be made to correlate the two sets at suitable spots within the text.

Introduction

A method of analysis which maximizes the expected cumulative military gain[1] due to a sequence of a predetermined number of bombing strikes has been obtained. The analytical method is a result of formulating the sequential bombing problem as a problem in sequential decision theory where the target damage function plays the role of the decision theory utility function. The significance of the sequential analysis is that each strike decision is based on maximizing the expected military gain due to all the strikes and takes into account all previous decisions and all available information about previous and present target states. Since it is possible that there is more than one way of obtaining information between strikes and there are different military costs associated with these different ways, the analysis includes the optimal sequential selection of the way in which information is acquired. The analysis uses Bayesian decision theory and assumes known weapon delivery and information error statistics.

The proposed method of analysis encompasses all three situations as described on page 1 of the proposal, "The Sequential Analysis of Cumulative Damage", No. ER70-R-2, July 1, 1969, i.e. the situations where there is

(1) complete knowledge of the state of the target;

(2) no knowledge of the state of the target other than what strikes have been ordered; and

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[1]The military gain is defined to be the target damage minus the military cost of obtaining information between bombing strikes. When there is no military cost associated with obtaining information, the military gain is just the target damage.

(3) information as to the state of the target but the information contains errors.

In the special case where no information about the state of the target will be available during the sequence of strikes (Situation 2), the sequential analysis degenerates into a nonsequential analysis. The decisions are made so as to maximize the expected military gain due to all the strikes, but since no new information will be available during the sequence of strikes, all the strike decisions can be made before the sequence starts.

Theorems 1 and 2 of Chapter II can be used to show, under reasonably general consitions, that when there is no military cost associated with obtaining information, the expected military gain in Situation 1 is at least as large as that in Situations 2 and 3. Also, the expected military gain in Situation 3 is at least as large as that in Situation 2. These conclusions agree with what we would expect intuitively.

Formulation and Solution

Suppose there is a military target with known reference location[2]. In accordance with Snow's (1966) development, the target immediate military value is denoted by W and is a function of the possible states of the target, 0,1,2,... . 1 is the initial state of the target and $W(1) = 1$. 0 is the useless state and $W(0) = 0$. The states $m = 2,3,\ldots$ are partial damage states and $0 < W(m) \leq 1$ if $m = 2,3,\ldots$. It is assumed here that the state of the target can be changed only by use of

[2] Only the certain location problem is considered here. Extension of the analysis to the more general case where the target is uncertain can be easily accomplished if the required error probability distribution is known.

a weapon against the target, i.e. the enemy will not change the state of the target between strikes. The case when this is not a valid assumption is considered later. Under this assumption it makes sense to suppose that a finite number, M, of bombing strikes will be made on the target.

In a general sense, the decisions as to how the M bombing strikes will be made involve a number of parameters. Examples are aircraft to be used, weapon, aircraft height, aircraft velocity, and desired ground zero (DGZ) for the weapon detonation. The point of view is taken here that all the pertinent parameters, with the exception of the desired ground zeros and weapons, are dictated by some conditions beyond the decision makers' control. Thus, each strike decision involves only the choice of DGZ and weapon for that strike. Denote the decision as to the DGZ for the j^{th} strike by $(\bar{x}_j,\bar{y}_j)$. Also denote the decision as to the weapon for the j^{th} strike by $\bar{b}_j$. It is assumed that there is a well defined set of choices for $(\bar{x}_j,\bar{y}_j)$ and $\bar{b}_j$ for each $j = 1,2,\ldots,M$. For $j \leq M$, the collection of decisions, $((\bar{x}_1,\bar{y}_1),\bar{b}_1;(\bar{x}_2,\bar{y}_2),\bar{b}_2;\ldots;(\bar{x}_j,\bar{y}_j),\bar{b}_j)$ will be denoted by $\underline{(\bar{x}_j,\bar{y}_j),\bar{b}_j}$.

$\omega_{m,n}((x_j,y_j),\bar{b}_j)$ denotes the probability that the j^{th} bomb will transfer the target to state n given that the target was in state m just before the j^{th} bomb was dropped, the bomb detonates at (x_j,y_j), and the bomb used was $\bar{b}_j$. For $j \leq M$, the collection of detonation points $((x_1,y_1);(x_2,y_2);\ldots;(x_j,y_j))$ will be denoted by $\underline{(x_j,y_j)}$. The weapon delivery error statistics are assumed known in the form of the conditional probability density function,

$$f_{\underline{(X_M,Y_M)}\,|\,\underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M}}(\underline{(x_M,y_M)}\,|\,\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}) \quad .$$

Let $\overline{\omega}_{m,n}((\overline{x}_j,\overline{y}_j),\overline{b}_j)$ denote the probability that the j^{th} bomb will transfer the target to state n given that the target was in state m just before the j^{th} bomb was dropped, the DGZ is $(\overline{x}_j,\overline{y}_j)$, and the bomb used was $\overline{b}_j$. This probability is given by the equation

$$\overline{\omega}_{m,n}((\overline{x}_j,\overline{y}_j),\overline{b}_j)$$

$$= \int \omega_{m,n}((x_j,y_j),\overline{b}_j)\ f_{(X_j,Y_j)|(\overline{X}_j,\overline{Y}_j),\overline{B}_j}((x_j,y_j)|(\overline{x}_j,\overline{y}_j),\overline{b}_j)d(x_j,y_j)$$

where it is assumed that

$$f_{(X_j,Y_j)|\underline{(\overline{X}_M,\overline{Y}_M),\overline{B}_M}}((x_j,y_j)|\underline{(\overline{x}_M,\overline{y}_M),\overline{b}_M})$$

$$= f_{(X_j,Y_j)|(\overline{X}_j,\overline{Y}_j),\overline{B}_j}((x_j,y_j)|(\overline{x}_j,\overline{y}_j),\overline{b}_j) \quad .$$

This assumption makes sense from a physical point of view since it seems unlikely that DGZs and weapon choices on strikes other than the j^{th} could effect the statistics of the outcome of where the j^{th} bomb falls given the j^{th} DGZ and $\overline{b}_j$.

It is assumed that the state of the target is Markov. Let Z_j be the random variable which is the state of the target just after the j^{th} strike. For $j \leq M$, let $\underline{Z}_j$ denote $(Z_1,Z_2,\ldots,Z_j)$. Let $f_{\underline{Z}_M|\underline{(\overline{X}_M,\overline{Y}_M),\overline{B}_M}}(i,j,\ldots,q|\underline{(\overline{x}_M,\overline{y}_M),\overline{b}_M})$ denote the probability that the target state after the 1^{st} strike is i, the target state after the 2^{nd} strike is j,..., the target state after the M^{th} strike is q given $\underline{(\overline{x}_M,\overline{y}_M),\overline{b}_M}$. Using the Markov assumption, this probability is given by

$$f_{\underline{Z}_M|\underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M}}(i,j,\ldots,p,q|\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M})$$

$$= \bar{\omega}_{1i}((\bar{x}_1,\bar{y}_1),\bar{b}_1) \quad \bar{\omega}_{ij}((\bar{x}_2,\bar{y}_2),\bar{b}_2) \quad \ldots \quad \bar{\omega}_{pq}((\bar{x}_M,\bar{y}_M),\bar{b}_M) \quad .$$

Notice that this probability mass function should be identified with the probability density function, $f_{\underline{X}_n|\underline{D}_n}$, which was introduced in the general formulation of Chapter II.

Between the $j-1^{th}$ and j^{th} strikes, information may be obtained as to the past and present states of the target. The set of ways in which information can be obtained is assumed to be well defined for each $j = 1,2,\ldots,M-1$. The specific choice as to the way information will be obtained in denoted by the variable k_{j-1}. Thus, k_{j-1} must belong to the set of ways in which information can be obtained. As a result of this choice, information will be obtained in the form of an observation on the random variable(s), $I^{k_{j-1}}$, at a military cost of $C(k_{j-1})$. The collection of ways in which information can be obtained might represent the ways in which an observation of the target area can be obtained. From an observation of the target area, an estimate of the state of the target could be obtained. The military cost associated with a particular way of observing the target area might represent an expected loss of aircraft, personnel, etc.

For $j \leq M-1$, the notation $I^{\underline{k}_j}$ will be used to indicate the collection $(I^{k_1}, I^{k_2}, \ldots, I^{k_j})$ and $\underline{k}_j$ will be used to indicate the collection $(k_1,k_2,\ldots,k_j)$. The information error statistics are assumed known in the form of the probability mass function,

$$f_{I^{\underline{k}_{M-1}}|\underline{(X_M,Y_M),B_M},\underline{Z}_M}(i^{\underline{k}_{M-1}}|\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}, i,j,\ldots,q)$$

which is the probability that $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$ given $\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}$, the target state after the 1^{st} strike is i, the target state after the 2^{nd} strike is j,..., the target state after the M^{th} strike is q. Notice that this probability mass function should be identified with the probability density function, $f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n}$, which was introduced in the general formulation of Chapter II. Usually this probability mass function would not be expected to depend on the condition $(\bar{x}_M,\bar{y}_M),\bar{b}_M$.

For $j < M-1$, marginal mass functions for $I^{\underline{k}_j}$ may be obtained from the known mass function. Physical reasoning leads to the assumption that the conditional density of observations $I^{\underline{k}_j}$ is independent of target states which have not occurred at the time of the observation of I^{k_j}.

The problem is to determine the sequential rules, $\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1},\tilde{k}_1$, $\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2},\tilde{k}_2,\ldots,\tilde{k}_{M-1},\widetilde{(\bar{x}_M,\bar{y}_M),\bar{b}_M}$ for the strike decisions and information choices so as to maximize the expectation of the military gain,

$$1-W(Z_M) - \sum_{j=1}^{M-1} C(k_j) \qquad .$$

Note that the damage, $1-W(Z_M)$, should be identified with the utility function, $V(\underline{x}_n,\underline{d}_n)$, which was introduced in the general formulation of Chapter II.

The solution to the sequential bombing problem can be obtained using the method of backward induction. The procedure is to work backward from the M^{th} strike decision, establishing optimal rules for each strike decision and information choice in terms of what is known at the time that decision or information choice is to be made.

Accordingly, the conditional expectation of the military gain given $\underline{(x_M,y_M),b_M}$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$ is computed and then maximized with respect to the decision, $(\bar{x}_M,\bar{y}_M),\bar{b}_M$. The required expectation may be written

$$\sum_{i,j,\ldots,q} [1-W(q)-\sum_{j=1}^{M-1} C(k_j)]\, f_{\underline{Z}_M|\underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M},I^{\underline{k}_{M-1}}}(i,j,\ldots,q|\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M},i^{\underline{k}_{M-1}})$$

where

$$f_{\underline{Z}_M|\underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M},I^{\underline{k}_{M-1}}}(i,j,\ldots,q|\underline{(\bar{x}_M,\bar{y}_M)},\bar{b}_M,i^{\underline{k}_{M-1}})$$

is the probability that the target state after the 1^{st} strike is i, the target state after the 2^{nd} strike is j,..., the target state after the M^{th} strike is q given $\underline{(\bar{x}_M,\bar{y}_M)},\bar{b}_M$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$. This probability can be found, using Bayes' rule, from the known probability mass functions.

$$f_{\underline{Z}_M \mid \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M},\, I^{\underline{k}_{M-1}}}\left(i,j,\ldots,q \mid \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M},\, i^{\underline{k}_{M-1}}\right)$$

$$= \left[f_{I^{\underline{k}_{M-1}} \mid \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M},\underline{Z}_M}\left(i^{\underline{k}_{M-1}} \mid \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M},i,j,\ldots,q\right) f_{\underline{Z}_M \mid \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M}}\left(i,j,\ldots,q \mid \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}\right) \right] \Bigg/$$

$$\left[\sum_{i,j,\ldots,q} f_{I^{\underline{k}_{M-1}} \mid \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M},\underline{Z}_M}\left(i^{\underline{k}_{M-1}} \mid \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M},i,j,\ldots,q\right) \right.$$

$$\left. f_{\underline{Z}_M \mid \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M}}\left(i,j,\ldots,q \mid \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}\right) \right] .$$

The choice for the M^{th} strike decision, $(\bar{x}_M,\bar{y}_M),\bar{b}_M$, which maximizes the conditional expectation of the military gain given $\underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$ defines the optimal M^{th} strike decision rule, $\widetilde{(\bar{x}_M,\bar{y}_M),\bar{b}_M}$. This rule is of course a function of $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-1}$, and $i^{\underline{k}_{M-1}}$. Substitution of the rule into the expectation yields the maximum expectation of the military gain given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$.

The next step is to compute the conditional expectation of the maximum expectation of the military gain given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$ given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $I^{\underline{k}_{M-2}} = i^{\underline{k}_{M-2}}$ and then maximize with respect to the information choice, k_{M-1}. The required expectation may be written

$$\sum_{i^{k_{M-1}}} f_{I^{k_{M-1}} | \underline{(\bar{X}_{M-1},\bar{Y}_{M-1}),\bar{B}_{M-1}}, I^{\underline{k}_{M-2}}} (i^{k_{M-1}} | \underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}, i^{\underline{k}_{M-2}})$$

$$\max_{(\bar{x}_M,\bar{y}_M),\bar{b}_M} \left[\sum_{i,j,\ldots,q} [1-W(q) - \sum_{j=1}^{M-1} C(k_j)] \right.$$

$$\left. f_{\underline{Z}_M | \underline{(\bar{X}_M,\bar{Y}_M),\bar{B}_M}, I^{\underline{k}_{M-1}}} (i,j,\ldots,q | \underline{(\bar{x}_M,\bar{y}_M),\bar{b}_M}, i^{\underline{k}_{M-1}} \right]$$

where

$$f_{I^{k_{M-1}} | \underline{(\bar{X}_{M-1},\bar{Y}_{M-1}),\bar{B}_{M-1}}, I^{\underline{k}_{M-2}}} (i^{k_{M-1}} | \underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}, i^{\underline{k}_{M-2}})$$

is the probability that $I^{k_{M-1}} = i^{k_{M-1}}$ given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $I^{\underline{k}_{M-2}} = i^{\underline{k}_{M-2}}$. This probability can be found from the equation,

$$f_{I^{\underline{k}_{M-1}} | \underline{(\bar{X}_{M-1},\bar{Y}_{M-1}),\bar{B}_{M-1}}, I^{\underline{k}_{M-2}}} (i^{k_{M-1}} | \underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}, i^{\underline{k}_{M-2}}) = \sum_{i,j,\ldots,p} \left[\right.$$

$$f_{I^{k_{M-1}} | \underline{(\bar{X}_{M-1},\bar{Y}_{M-1}),\bar{B}_{M-1}}, I^{\underline{k}_{M-2}}, \underline{Z}_{M-1}} (i^{k_{M-1}} | \underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}, i^{\underline{k}_{M-2}}, i,j,\ldots,p)$$

$$\left. f_{\underline{Z}_{M-1} | \underline{(\bar{X}_{M-1},\bar{Y}_{M-1}),\bar{B}_{M-1}}, I^{\underline{k}_{M-2}}} (i,j,\ldots,p | \underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}, i^{\underline{k}_{M-2}}) \right] .$$

The choice for the M-1th way of obtaining information, k_{M-1}, which maximizes the conditional expectation of the maximum expectation of the military gain given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-1}$, and $I^{\underline{k}_{M-1}} = i^{\underline{k}_{M-1}}$ given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $I^{\underline{k}_{M-2}} = i^{\underline{k}_{M-2}}$ defines the optimal M-1th information choice rule, $\tilde{k}_{M-1}$. This rule is of course a function of $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $i^{\underline{k}_{M-2}}$. Substitution of the rule into the expectation yields the maximum expectation of the military gain given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $I^{\underline{k}_{M-2}} = i^{\underline{k}_{M-2}}$.

The next step is to maximize the maximum expectation of the military gain given $\underline{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$, $\underline{k}_{M-2}$, and $I^{\underline{k}_{M-2}} = i^{\underline{k}_{M-2}}$ with respect to the decision, $(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}$. This maximization determines the optimal M-1th strike decision rule, $\widetilde{(\bar{x}_{M-1},\bar{y}_{M-1}),\bar{b}_{M-1}}$.

Continuing to work backward in this fashion, the complete set of optimal strike decision and information choice rules may be obtained. As a side product of the solution procedure, the value for the maximum expected cumulative military gain is also obtained. The solution procedure is illustrated in the following example.

Example

Suppose there is a military target which has the possible states, 0, 1, and 2. If the target is in state 2, its military value is $\frac{1}{2}$. There will be two bombing strikes made on the target. The same two choices of weapon will be available for both strikes. The two choices will be denoted by A and D. The transition probability mass function, $\omega_{m,n}((x_j,y_j),\bar{b}_j)$, and the weapon delivery error statistics are such that the transition probability mass function, $\bar{\omega}_{m,n}((\bar{x}_j,\bar{y}_j),\bar{b}_j)$, is the function expressed by the listing below:

$$\bar{\omega}_{1,0}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = 0$$

$$\bar{\omega}_{2,1}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = 0$$

$$\bar{\omega}_{0,2}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = 0$$

$$\bar{\omega}_{0,1}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = 0$$

$$\bar{\omega}_{0,0}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = 1$$

$$\bar{\omega}_{1,1}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = \begin{cases} 1 \text{ , if } \bar{b}_j = A \\ \frac{1}{4} \text{ , if } \bar{b}_j = D,\ 1 < \bar{x}_j^2 + \bar{y}_j^2 \leq 2 \\ \frac{3}{4} \text{ , if } \bar{b}_j = D,\ \bar{x}_j^2 + \bar{y}_j^2 \leq 1 \\ \frac{3}{4} \text{ , if } \bar{b}_j = D,\ 2 < \bar{x}_j^2 + \bar{y}_j^2 \end{cases}$$

$$\bar{\omega}_{2,2}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = \begin{cases} 1 \text{ , if } \bar{b}_j = D \\ \frac{1}{4} \text{ , if } \bar{b}_j = A,\ \bar{x}_j^2 + \bar{y}_j^2 \leq 1 \\ \frac{3}{4} \text{ , if } \bar{b}_j = A,\ 1 < \bar{x}_j^2 + \bar{y}_j^2 \leq 2 \\ 1 \text{ , if } \bar{b}_j = A,\ 2 < \bar{x}_j^2 + \bar{y}_j^2 \end{cases}$$

$$\bar{\omega}_{1,2}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = \begin{cases} 0 \text{ , if } \bar{b}_j = A \\ \frac{3}{4} \text{ , if } \bar{b}_j = D, \ 1 < \bar{x}_j^2 + \bar{y}_j^2 \leq 2 \\ \frac{1}{4} \text{ , if } \bar{b}_j = D, \quad \bar{x}_j^2 + \bar{y}_j^2 \leq 1 \\ \frac{1}{4} \text{ , if } \bar{b}_j = D, \quad 2 < \bar{x}_j^2 + \bar{y}_j^2 \end{cases}$$

$$\bar{\omega}_{2,0}((\bar{x}_j,\bar{y}_j),\bar{b}_j) = \begin{cases} 0 \text{ , if } \bar{b}_j = D \\ \frac{3}{4} \text{ , if } \bar{b}_j = A, \quad \bar{x}_j^2 + \bar{y}_j^2 \leq 1 \\ \frac{1}{4} \text{ , if } \bar{b}_j = A, \ 1 < \bar{x}_j^2 + \bar{y}_j^2 \leq 2 \\ 0 \text{ , if } \bar{b}_j = A, \quad 2 < \bar{x}_j^2 + \bar{y}_j^2 \end{cases} .$$

The strike decision and information choice rules and the maximum expected military gain will be obtained in the following 4 situations.

(1) The set of ways to obtain information contains the three ways α, β, and γ and $C(\alpha) = C(\beta) = C(\gamma) = 0$.

(2) The set of ways to obtain information contains the one way α and $C(\alpha) = 0$.

(3) The set of ways to obtain information contains the two ways α and β and $C(\alpha) = C(\beta) = 0$.

(4) The set of ways to obtain information contains the three ways

α, β, and γ and $C(\alpha) = 0$, $C(\beta) = .04$, $C(\gamma) = .09$, where:

$$f_{I^{\alpha}|(\overline{X}_1,\overline{Y}_1),\overline{B}_1,Z_1}(x|(\overline{x}_1,\overline{y}_1),\overline{b}_1,p) = \begin{cases} 1 , & x = i^{\alpha} \\ 0 , & \text{otherwise} \end{cases}$$

and i^{α} is a known constant

$$f_{I^{\beta}|(\overline{X}_1,\overline{Y}_1),\overline{B}_1,Z_1}(p|(\overline{x}_1,\overline{y}_1),\overline{b}_1,q) = \begin{cases} .8 , & \text{if } p = q \\ .1 , & \text{if } p \neq q \end{cases}$$

$$f_{I^{\gamma}|(\overline{X}_1,\overline{Y}_1),\overline{B}_1,Z_1}(p|(\overline{x}_1,\overline{y}_1),\overline{b}_1,q) = \begin{cases} 1 & , \text{if } p = q \\ 0 & , \text{if } p \neq q \end{cases} .$$

Notice that since $I^{\alpha} = i^{\alpha}$ with a probability of one regardless of the target state Z_1, α is a way of obtaining information which yields no information about Z_1. α is equivalent to obtaining no information at all. β is a way of obtaining information which does yield information about the state of the target, but the information may be in error. γ is a way of obtaining information about the state of the target which is complete or exact.

<u>Situation 3</u>

Since neither α or β have a cost associated with them, it may be argued heuristically that β is the optimal information choice, i.e. $\tilde{k}_1 = \beta$. Theorem 1 of Chapter II can be used to establish that this is the case. The present situation is the same as Situation 3 mentioned in the introduction of this chapter.

The first step in obtaining the solution is to use Bayes' rule to obtain the probability mass function, $f_{Z_2|\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^\beta}(i,j|(\bar{x}_2,\bar{y}_2),\bar{b}_2,i^\beta)$, compute the conditional expectation of $1-W(Z_2)$ given $\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$, β, and $I^\beta = i^\beta$, and by direct comparison determine the 2^{nd} strike decision rule, $\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$. Substitution of the rule into the expectation then yields the maximum expectation of the damage given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$, β, and $I^\beta = i^\beta$. The arithmetic has been carried out and the results are given in Table I.

Next, the probability mass function, $f_{I^\beta|\underline{(\bar{X}_1,\bar{Y}_1),\bar{B}_1}}(i^\beta|\underline{(\bar{x}_1,\bar{y}_1),\bar{b}_1})$, is obtained and used to compute the conditional expectation of the maximum expectation of the damage given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$, β, and $I^\beta = i^\beta$, given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$. Since $\tilde{k}_1 = \beta$, this result is the maximum expectation of the damage given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$. The results are given in Table II.

By comparison of the results in Table II, the first strike decision rule and the maximum expected damage can be obtained.

$$\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1} = 1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2, \ D$$

$$\text{Max. Expected Damage} = \frac{22.5}{32} \cong .703 \qquad .$$

Knowing what the 1^{st} strike decision is provides some

TABLE I

SITUATION 3: SECOND STRIKE DECISION RULE AND MAXIMUM EXPECTED DAMAGE GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$, β, and $I^\beta = i^\beta$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	I^β	$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$	$\max_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum_{i,j} [1-W(j)] f_{\underline{Z}_2 \mid \underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^\beta}(i,j \mid \underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^\beta)$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \le 2$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \le 2$, D	3/8
"	"	1	" "	"
"	"	2	" "	"
"	$\bar{x}_1^2 + \bar{y}_1^2 \le 1$	0	" "	"
"	"	1	" "	"
"	"	2	" "	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	0	" "	"
"	"	1	" "	"
"	"	2	" "	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \le 2$	0	$\bar{x}_2^2 + \bar{y}_2^2 \le 1$, A	21/32
"	"	1	$1 < \bar{x}_2^2 + \bar{y}_2^2 \le 2$, D	[(3/8)(.8)+(3/2)(.1)]/[.3+.8]
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \le 1$, A	[(21/8)(.8)]/[2.4+.1]

TABLE I (Continued)

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	I^β	$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$	$\max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} [1-W(j)]f_{\underline{Z}_2 \mid \underline{(\bar{X}_2,\bar{Y}_2)},\underline{\bar{B}_2},I^\beta}(i,j \mid \underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^\beta)$
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	13/32
"	"	1	" "	[(9.8)(.8)+(1/2)(.1)]/[.1+2.4]
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \leq 1$, A	[(7/8)(.8)]/[.8+.3]
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	13/32
"	"	1	" "	[(9.8)(.8)+(1/2)(.1)]/[.1+2.4]
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \leq 1$, A	[(7/8)(.8)]/[.8+.3]

TABLE II

SITUATION 3: MAXIMUM EXPECTED DAMAGE GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	$\max_{k_1} \sum_{i^{k_1}} f_{I^{k_1}\|(\bar{X}_1,\bar{Y}_1),\bar{B}_1}(i^{k_1}\|(\bar{x}_1,\bar{y}_1),\bar{b}_1) \max_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum_{i,j} [1-W(j)] f_{\underline{Z}_2\|\underline{(\bar{X}_2,\bar{Y}_2)},\bar{B}_2,I^{k_1}}(i,j\|\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^{k_1})$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	3/8
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	22.5/32
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	14.5/32
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	14.5/32

simplification for the 2nd strike decision rule. The strike decision and information choice rules are summarized below.

$$\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1} = 1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2, \text{ D}$$

$$\tilde{k}_1 = \beta$$

$$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2} = \begin{cases} \bar{x}_2^2 + \bar{y}_2^2 \leq 1 & \text{, A if } I^\beta = 0 \\ 1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2, & \text{D if } I^\beta = 1 \\ \bar{x}_2^2 + \bar{y}_2^2 \leq 1 & \text{, A if } I^\beta = 2 \end{cases} \quad .$$

Situation 1

In this situation, heuristic reasoning indicates that γ is the optimal choice for k_1 and Theorem 2 of Chapter II can be used to establish that this is the case. The present situation is the same as Situation 1 mentioned in the introduction of this chapter.

The procedure for solution is the same as in Situation 3. The results for the present situation which correspond to the results given in Tables I and II for Situation 3 are given in Tables III and IV.

By comparison of the results in Table IV, the first strike decision rule and the maximum expected damage is obtained.

$$\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1} = 1 < \bar{x}_1 + \bar{y}_1 \leq 2, \text{ D}$$

$$\text{Max. Expected Damage} = \frac{24}{32} = .750 \quad .$$

TABLE III

SITUATION 1: SECOND STRIKE DECISION RULE AND MAXIMUM EXPECTED DAMAGE GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$, γ, and $I^\gamma = i^\gamma$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	I^γ	$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$	$\max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} [1-W(j)]f_{\underline{Z}_2\|\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^\gamma}(i,j\|\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^\gamma)$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \le 2$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \le 2$, D	3/8
"	"	1	" "	"
"	"	2	" "	"
"	$\bar{x}_1^2 + \bar{y}_1^2 \le 1$	0	" "	"
"	"	1	" "	"
"	"	2	" "	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	0	" "	"
"	"	1	" "	"
"	"	2	" "	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \le 2$	0	$\bar{x}_2^2 + \bar{y}_2^2 \le 1$, A	21/32
"	"	1	$1 < \bar{x}_2^2 + \bar{y}_2^2 \le 2$, D	3/8
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \le 1$, A	21/24

TABLE III (Continued)

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	I^{γ}	$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$	$\max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} [1-W(j)]f_{\underline{Z}_2\mid\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^{\gamma}}(i,j\mid\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^{\gamma})$
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	13/32
"	"	1	" "	9/24
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \leq 1$, A	7/8
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	0	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	13/32
"	"	1	" "	9/24
"	"	2	$\bar{x}_2^2 + \bar{y}_2^2 \leq 1$, A	7/8

TABLE IV

SITUATION 1: MAXIMUM EXPECTED DAMAGE GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	$\max\limits_{k_1} \sum\limits_{i^{k_1}} f_{I^{k_1}\|(\bar{X}_1,\bar{Y}_1),\bar{B}_1}(i^{k_1}\|(\bar{x}_1,\bar{y}_1),\bar{b}_1) \max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} \boxed{1-W(j)} f_{\underline{Z}_2\|\underline{(\bar{X}_2,\bar{Y}_2)},\bar{B}_2,I^{k_1}}(i,j\|\underline{(\bar{x}_2,\bar{y}_2)},\underline{\bar{b}_2},i^{k_1})$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	3/8
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	24/32
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	16/32
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	16/32

The strike decision and information choice rules are summarized below.

$$\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1} = 1 < \bar{x}_1^2 + \bar{y}_1^2 \le 2,\ D$$

$$\tilde{k}_1 = \gamma$$

$$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2} = \begin{cases} \bar{x}_2^2 + \bar{y}_2^2 \le 1 & ,\ A \text{ if } I^\gamma = 0 \\ 1 < \bar{x}_2^2 + \bar{y}_2^2 \le 2, & D \text{ if } I^\gamma = 1 \\ \bar{x}_2^2 + \bar{y}_2^2 \le 1 & ,\ A \text{ if } I^\gamma = 2 \end{cases} .$$

<u>Situation 2</u>

In this situation, since the only way of obtaining information is α, $\tilde{k}_1 = \alpha$. Formally, the solution proceeds in exactly the same way as in Situations 1 and 3; however, since I^α is a degenerate random variable,

$$f_{\underline{Z}_2|\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^\alpha}(i,j|\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},x) = f_{\underline{Z}_2|\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2}}(i,j|\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2}) .$$

Thus, to obtain the solution all that is necessary is to obtain the expectation of the damage given $\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2}$ and then maximize with to $(\bar{x}_2,\bar{y}_2),\bar{b}_2$ and $(\bar{x}_1,\bar{y}_1),\bar{b}_1$ to obtain the strike decisions. Because the results will be needed in the solution of Situation 4, the maximization with respect to $(\bar{x}_2,\bar{y}_2),\bar{b}_2$ was carried out first and the 2nd strike decisions and the maximum expected damage given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$ are tabulated in Table V.

By comparison of the results in Table V, the first strike decision

TABLE V

SITUATION 2: SECOND STRIKE DECISION RULE AND MAXIMUM EXPECTED DAMAGE GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	$\widetilde{(\bar{x}_2,\bar{y}_2)},\bar{b}_2$	$\max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} [1-W(j)]f_{\underline{Z}_2 \mid \underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2}}(i,j \mid \underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2})$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	3/8
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	" "	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	" "	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	$\bar{x}_2^2 + \bar{y}_2^2 \leq 1$, A	21/32
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	$1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2$, D	13/32
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	" "	"

rule and the maximum expected damage is obtained.

$$\widetilde{(\bar{x}_1,\bar{y}_1),\bar{b}_1} = 1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2,\ D$$

$$\text{Max. Expected Damage} = \frac{21}{32} \cong .656 \quad .$$

The second strike decision rule is of course

$$\widetilde{(\bar{x}_2,\bar{y}_2),\bar{b}_2} = \bar{x}_2^2 + \bar{y}_2^2 \leq 1,\ A$$

Notice that since there is no new information obtained between strikes, both strike decisions can be made before the sequence starts.

The maximum expected damage in Situation 1 is greater than in Situation 2 and 3. Also, the maximum expected damage in Situation 3 is greater than in Situation 2. These results agree with what would be expected intuitively.

<u>Situation 4</u>

The solution to Situation 4 may be obtained by using the results obtained in Situations 1, 2, and 3. First observe that the 2^{nd} strike decision rules tabulated in Tables I, III, and V define the 2^{nd} strike decision rule given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$, k_1 and $I^{k_1} = i^{k_1}$. The information choice rule, $\tilde{k}_1$, may be obtained by comparing the right most column of Tables II, IV, and V, taking into account the cost, $C(k_1)$, of the information in each table. The rule, the cost function C, and the results of Tables II, IV, and V can be used to obtain the maximum expectation of the military gain given $(\bar{x}_1,\bar{y}_1),\bar{b}_1$. The results are given in Table VI.

TABLE VI

SITUATION 4: INFORMATION CHOICE RULE AND MAXIMUM EXPECTED MILITARY GAIN GIVEN $(\bar{x}_1,\bar{y}_1),\bar{b}_1$

$\bar{b}_1$	$(\bar{x}_1,\bar{y}_1)$	$\tilde{k}_1$	$\max\limits_{k_1} \sum\limits_{i^{k_1}} f_{I^{k_1}\mid(\bar{X}_1,\bar{Y}_1),\bar{B}_1}(i^{k_1}\mid(\bar{x}_1,\bar{y}_1),\bar{b}_1) \max\limits_{(\bar{x}_2,\bar{y}_2),\bar{b}_2} \sum\limits_{i,j} [1-W(j)-C(k_1)] f_{\underline{Z}_2\mid\underline{(\bar{X}_2,\bar{Y}_2),\bar{B}_2},I^{k_1}}(i,j\mid\underline{(\bar{x}_2,\bar{y}_2),\bar{b}_2},i^{k_1})$
A	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	α	3/8
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	"	"
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	"	"
D	$1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2$	β	.663
"	$\bar{x}_1^2 + \bar{y}_1^2 \leq 1$	"	.413
"	$2 < \bar{x}_1^2 + \bar{y}_1^2$	"	"

By simple comparison of the results in Table VI, the first strike decision rule and the maximum expected military gain can be obtained. Knowledge of the first strike decision provides some simplification for the information choice and second strike decision rules. The strike decision and information choice rules and the maximum expected military gain are summarized below.

$$\widetilde{(\bar{x}_1, \bar{y}_1), \bar{b}_1} = 1 < \bar{x}_1^2 + \bar{y}_1^2 \leq 2,\ D$$

$$\tilde{k}_1 = \beta$$

$$\widetilde{(\bar{x}_2, \bar{y}_2), \bar{b}_2} = \begin{cases} \bar{x}_2^2 + \bar{y}_2^2 \leq 1 & ,\ A \text{ if } I^\beta = 0 \\ 1 < \bar{x}_2^2 + \bar{y}_2^2 \leq 2, & D \text{ if } I^\beta = 1 \\ \bar{x}_2^2 + \bar{y}_2^2 \leq 1 & ,\ A \text{ if } I^\beta = 2 \end{cases}$$

Max. Expected Military Gain = .663 .

When there was no military cost associated with obtaining information, exact information (Situation 1) proved to be better than information with possible error (Situation 3). The present situation represents a case where the military cost associated with obtaining exact information makes obtaining information with possible error the better strategy.

Determination of M, the Number of Strikes

In the preceding work it was assumed that the number of bombing strikes, M, was specified as part of the sequential bombing problem. Since there has been no cost associated with making a bombing strike, the strategist could merely choose M large enough so as to insure that the maximum expected military gain would be large enough. This is obviously an impractical solution which ignores many limiting factors and defeats the idea of obtaining a satisfactory military gain with as few strikes as possible. Thus, a subjective decision as to the number of strikes may not be adequate and some systematic analytical means of choosing M is needed.

There are at least two approaches to the problem.

(1) The sequential bombing problem can be solved for each $M = 1, 2, \ldots$ until the resulting maximum expected military gain is sufficiently close to 1.

(2) Using an educated guess as to the optimal strike decision and experiment choice rules, obtain an approximate answer for the maximum expected military gain for each $M = 1, 2, \ldots$. The process would be stopped and an exact solution obtained when a value of M was found such that the approximate maximum expected military gain was sufficiently close to 1.

Both approaches depend on a subjective judgement as to how large "sufficiently close to 1" is, but an expected military gain between .9 and 1.0 would seem reasonable. Obtaining a fairly accurate approximate answer for the maximum expected military gain as suggested under (2) above is, in many cases, not as difficult as the reader might expect. A little thought about Situations 1, 2, and 3 of the previous example and the optimal strike decision and information choice rules can be written down. Obtaining the approximate maximum expected military gain for a given M with the approximate rules specified is much less work than obtaining the maximum expected military gain by the optimal solution procedure. Thus, approach (1) would probably require much more computation time than approach (2), while approach (2) might result in a larger strike number than necessary if bad guesses were made as to the approximate strike decision and information choice rules.

Target Modification Between Strikes

In the preceding work it was assumed that the only way the state of the target could be changed was by using a weapon against the target. Under this assumption it was appropriate to suppose a finite number of strikes, M, would be made on the target and to use as the criteria for success of the M strikes the military gain, $1-W(Z_M) - \sum_{j=1}^{M-1} C(k_j)$.

Suppose now that the state of the target may be changed between strikes and that $\omega'_{m,n}$ is a probability mass function which is the probability that the target will be transferred to state n given that the last strike left the target in state m. Under such conditions it may be appropriate to suppose that a very large number of strikes will be made on the target and to use as the criteria for success, the military gain per strike. The concept of an infinite sequence of strikes will be used to approximate the idea of a very large number of strikes.

Under the assumptions which will be made, it can be assumed that the states of the target just before and just after a strike have a time stationary probability distribution. Let Z' denote the random variable which is the state of the target just before a strike and Z denote the random variable which is the state of the target just after the strike. Suppose there is a well defined set of ways to obtain information, about the state Z', between strikes. The choice of way is denoted by k and results in the observation of a sample of the same random variable, I^k, between each strike at a military cost of C(k) per strike. The information error statistics are assumed known in the form of the probability mass function, $f_{I^k|Z'}(i^k|p)$, which is assumed to be independent of strike decisions and target states other than the state of the target, Z', which exists at the time the sample of I^k is obtained.

Each strike decision is to be chosen from the same well defined set of choices. $\overline{\omega}_{m,n}((\overline{x},\overline{y}),\overline{b})$ denotes the probability mass function which is the probability that the state of the target will be transferred to state n given that the target was in state m just before the strike and the strike decision $(\overline{x},\overline{y}),\overline{b}$ was made.

Suppose now that an information choice, k, has been made. It would be feasible to continually retain any fixed finite number of past strike decisions and samples of I^k and to make the strike decision rule a function of the stored data. There are at least two good arguments for restricting the strike decision rule to be a function only of the most current sample of I^k. First of all, except in special cases the information content in other data tends to be small because of the random changes in target state that occur after that data is obtained. Second, it is easy to solve for the time stationary state probability distributions which result from this restriction. For these reasons, the strike decision rule is restricted to be a function only of the most current sample of I^k. It is assumed that a sample of I^k is available at the time the 1^{st} strike decision must be made and that this is all that is known about the state of the target at that time.

Let $[(\bar{x},\bar{y}),\bar{b}](i^k)$ be a strike decision rule for the information choice, k. $\bar{\omega}_{m,n}([(\bar{x},\bar{y}),\bar{b}](i^k))$ is the probability mass function which is the probability that the state of the target will be transferred to state n given that the target was in state m just before the strike and the information i^k was obtained. Let $\omega''_{m,n}$ be the probability mass function which is the probability that the state of the target will be transferred to state n given that the target was in state m just before the strike. $\omega''_{m,n}$ is given by the equation,

$$\omega''_{m,n} = \sum_{i^k} \bar{\omega}_{m,n}\left([(\bar{x},\bar{y}),\bar{b}](i^k)\right) f_{I^k|Z'}(i^k|m) \quad .$$

Let Ω'' be the matrix whose i,jth element is $\omega''_{i,j}$. Also, let Ω' be the matrix whose i,jth element is $\omega'_{i,j}$. Then, the matrix $\Gamma = [\gamma_{i,j}] = \Omega''\Omega'$ is the transition matrix for Z'. Let $\Gamma^n = [\gamma_{i,j}]^n = [\gamma^n_{i,j}]$. If for each i and j, $\gamma^n_{i,j} > 0$ for some n, Z' has a stationary distribution, $f_{Z'}(p)$, which is independent of the starting state (Prabhu, 1965). In the work to follow, this is assumed to be the case.

The stationary distribution, $f_{Z'}(p)$, can be found by solving the system of equations,

$$\begin{bmatrix} f_{Z'}(0) \\ f_{Z'}(1) \\ \vdots \end{bmatrix} = \Gamma \begin{bmatrix} f_{Z'}(0) \\ f_{Z'}(1) \\ \vdots \end{bmatrix}$$

and

$$\sum_{p} f_{Z'}(p) = 1 \quad .$$

Although this stationary distribution may not exist initially, it will be approached as the number of completed strikes increases.

The expected military gain per strike can be found from the equation,

Expected Military Gain Per Strike

$$= \sum_{p} f_{Z'}(p) \left[\sum_{q} [W(p) - W(q) - C(k)]\omega''_{p,q} \right] \quad .$$

Within the imposed constraint on the strike decision rule, the optimal strike decision and information choice rules and the maximum expected military gain per strike are found by maximizing the expected military gain per strike first with respect to the strike decision rule and then with respect to the information choice. Thus,

Max. Expected Military Gain Per Strike

$$= \max_{k} \left[\max_{\text{strike rule}} \left[\sum_{p} f_{Z'}(p) \left[\sum_{q} [W(p) - W(q) - C(k)]\omega''_{p,q} \right] \right] \right] .$$

The rules thus obtained may not be optimal initially, but as the number of completed strikes becomes large, they will obtain at least as large a military gain per strike as any of the possible rules.

Example

Suppose there is a target with three possible states, 0, 1, and 2. The value of the target in state 2 is $\frac{1}{2}$. The matrix Ω' is

$$\Omega' = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} .$$

There is only one way of obtaining information. The one way is denoted α. The cost per strike, $C(\alpha)$, is zero. The information error statistics are expressed by the probability mass function,

$$f_{I^\alpha|Z'}(p|q) = \begin{cases} .8 , \text{ if } p = q \\ .1 , \text{ if } p \neq q \end{cases} .$$

There is only one choice of weapon for each strike and the DGZ must be chosen in regions A or A'. $\overline{\omega}_{m,n}$ is the function,

$$\overline{\omega}_{1,0}((\overline{x},\overline{y})) = 0$$

$$\overline{\omega}_{2,1}((\overline{x},\overline{y})) = 0$$

$$\overline{\omega}_{0,2}((\overline{x},\overline{y})) = 0$$

$$\overline{\omega}_{0,1}((\overline{x},\overline{y})) = 0$$

$$\overline{\omega}_{0,0}((\overline{x},\overline{y})) = 1$$

$$\bar{\omega}_{1,1}((\bar{x},\bar{y})) = \begin{cases} \frac{1}{2}, & \text{if } (\bar{x},\bar{y}) \varepsilon A \\ \frac{1}{4}, & \text{if } (\bar{x},\bar{y}) \varepsilon A' \end{cases}$$

$$\bar{\omega}_{1,2}((\bar{x},\bar{y})) = \begin{cases} \frac{1}{2}, & \text{if } (\bar{x},\bar{y}) \varepsilon A \\ \frac{3}{4}, & \text{if } (\bar{x},\bar{y}) \varepsilon A' \end{cases}$$

$$\bar{\omega}_{2,2}((\bar{x},\bar{y})) = \begin{cases} \frac{1}{4}, & \text{if } (\bar{x},\bar{y}) \varepsilon A \\ \frac{1}{2}, & \text{if } (\bar{x},\bar{y}) \varepsilon A' \end{cases}$$

$$\bar{\omega}_{2,0}((\bar{x},\bar{y})) = \begin{cases} \frac{3}{4}, & \text{if } (\bar{x},\bar{y}) \varepsilon A \\ \frac{1}{2}, & \text{if } (\bar{x},\bar{y}) \varepsilon A' \end{cases}$$

The 8 possible strike decision rules are listed below:

$$[(\bar{x},\bar{y})]_1(p) = \begin{cases} A, & \text{if } p = 1 \\ A, & \text{if } p = 2 \\ A, & \text{if } p = 0 \end{cases}$$

$$[(\overline{x},\overline{y})]_2(p) = \begin{cases} A & \text{, if } p = 1 \\ A & \text{, if } p = 2 \\ A' & \text{, if } p = 0 \end{cases}$$

$$[(\overline{x},\overline{y})]_3(p) = \begin{cases} A & \text{, if } p = 1 \\ A' & \text{, if } p = 2 \\ A & \text{, if } p = 0 \end{cases}$$

$$[(\overline{x},\overline{y})]_4(p) = \begin{cases} A & \text{, if } p = 1 \\ A' & \text{, if } p = 2 \\ A' & \text{, if } p = 0 \end{cases}$$

$$[(\overline{x},\overline{y})]_5(p) = \begin{cases} A' & \text{, if } p = 1 \\ A & \text{, if } p = 2 \\ A & \text{, if } p = 0 \end{cases}$$

$$[(\overline{x},\overline{y})]_6(p) = \begin{cases} A' & \text{, if } p = 1 \\ A & \text{, if } p = 2 \\ A' & \text{, if } p = 0 \end{cases}$$

$$[(x,y)]_7(p) = \begin{cases} A' & \text{, if } p = 1 \\ A' & \text{, if } p = 2 \\ A & \text{, if } p = 0 \end{cases}$$

$$[(x,y)]_8(p) = \begin{cases} A' & \text{, if } p = 1 \\ A' & \text{, if } p = 2 \\ A' & \text{, if } p = 0 \end{cases} \quad .$$

The matrix Ω'' is first found for each possible strike decision rule and then used to compute the matrix Γ. The results are given below.

$$\Omega''_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & 0 & \frac{1}{4} \end{bmatrix} \qquad \Omega''_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.9}{4} & \frac{2.1}{4} \\ \frac{2.9}{4} & 0 & \frac{1.1}{4} \end{bmatrix}$$

$$\Omega''_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.9}{4} & \frac{2.1}{4} \\ \frac{2.2}{4} & 0 & \frac{1.8}{4} \end{bmatrix} \qquad \Omega''_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.8}{4} & \frac{2.2}{4} \\ \frac{2.1}{4} & 0 & \frac{1.9}{4} \end{bmatrix}$$

$$\Omega''_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.2}{4} & \frac{2.8}{4} \\ \frac{2.9}{4} & 0 & \frac{1.1}{4} \end{bmatrix} \qquad \Omega''_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.1}{4} & \frac{2.9}{4} \\ \frac{2.8}{4} & 0 & \frac{1.2}{4} \end{bmatrix}$$

$$\Omega''_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1.1}{4} & \frac{2.9}{4} \\ \frac{2.1}{4} & 0 & \frac{1.9}{4} \end{bmatrix} \qquad \Omega''_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$\Gamma_1 = \Gamma_2 = \Gamma_3 = \ldots = \Gamma_8 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} .$$

The stationary distribution, $f_{Z'}(p)$, is the same for each possible strike decision rule.

$$f_{Z'}(p) = \frac{1}{3} \text{ , for } p = 0, 1, \text{ or } 2 \qquad .$$

Next, the expected military gain per strike or, in this example, the expected damage per strike is computed for each possible strike decision rule.

Rule 1: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5}{8})$

Rule 2: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5}{8})$

Rule 3: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{4.3}{8})$

Rule 4: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{4.3}{8})$

Rule 5: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5.7}{8})$

Rule 6: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5.7}{8})$

Rule 7: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5}{8})$

Rule 8: Expected Damage Per Strike = $\frac{1}{3}\ (\frac{5}{8})$.

Thus, for the information choice, α, the decision rule should be 5 or 6 and the expected damager per strike is $\frac{1}{3}\ (\frac{5.7}{8}) \cong .237$. Since α is the only way of obtaining information, the problem is completed.

CHAPTER V

APPLICATION TO SYSTEMS ENGINEERING

This chapter consists of a consideration of a systems engineering problem which has been discussed by Howard (1965).

A manufacturer is offered a fixed price contract to build and maintain a system of N devices for a period of T years. Every failure in the system during the T years must be replaced by the manufacturer at a cost of Z dollars. The system will cost J_0 dollars to establish, and the price of the contract is α. The manufacturer believes that the failures will be Poisson distributed with some rate X_1, but he is unsure about the value of X_1. The manufacturer does, however, obtain the probability density function for X_1, f_{X_1}. The problem the manufacturer is faced with is that of making the decision to accept or reject the contract and he would like to make that decision in such a way as to maximize his expected profit.

By proper interpretation of this problem, it can be made to fit the general problem formulation of Chapter II. Results identical with those obtained by Howard will then be obtained using the techniques developed in Chapter II.

Let X_2 represent the manufacturer's profit. Also, let $f_{X_1|D_1} = f_{X_1}$ and $\lambda_1 = \{\tilde{d}_1\}$, a singleton set. In other words, $f_{X_1|D_1}$ depends on the decision, d_1, but there is only one choice for that decision. λ_2 is a set containing two elements, a_2 and r_2, where a_2 is the decision to accept the contract and r_2 is the decision to reject the contract. As usual, ω_1 represents the collection of experiments which may be used to gain knowledge about X_1 before making the decision, d_2. The cost associated with the particular experiment choice, k_1, is as usual denoted by $C(k_1)$. The process of transforming the problem into familiar notation is completed by letting $V_1(\underline{x}_1,\underline{d}_1) = 0$ and $V_2(\underline{x}_2,\underline{d}_2) = x_2$.

The first step in obtaining the solution is to express the expectation of V_2 given $\underline{d}_2$, k_1, and $I^{k_1} = i^{k_1}$ in a form which is easier to work with. The expectation can be written,

$$\int_{\Omega_{\underline{X}_2}} V_2(\underline{x}_2,\underline{d}_2) f(|)_{\underline{X}_2|\underline{D}_2,I^{k_1}} d\underline{x}_2 = \int_{\Omega_{\underline{X}_2}} x_2 \, f(|)_{\underline{X}_2|\underline{D}_2,I^{k_1}} d\underline{x}_2$$

$$= \int_{\Omega_{X_1}} E(X_2|x_1,\underline{d}_2,i^{k_1}) \, f(|)_{X_1|\underline{D}_2,I^{k_1}} dx_1$$

$$= \int_{\Omega_{X_1}} E(X_2|x_1,d_2,i^{k_1}) \, f(|)_{X_1|I^{k_1}} dx_1$$

where $E(X_2|x_1,d_2,i^{k_1})$ denotes the expectation of the profit, X_2, given the failure rate $X_1 = x_1$, d_2, k_1, and $I^{k_1} = i^{k_1}$. d_1 has been dropped from the expressions since in this example, the first decision is ineffective and has no physical significance.

Using the fact that the failures are Poisson distributed with rate X_1, it is easy to obtain the result,

$$E(X_2|x_1,d_2,i^{k_1}) = \begin{cases} \alpha - J_0 - ZNT\,x_1 \;, & \text{if } d_2 = a_2 \\ 0 \;, & \text{if } d_2 = r_2 \end{cases} .$$

Thus, it is easy to rewrite the expectation of V_2 given d_2, k_1, and $I^{k_1} = i^{k_1}$ in the form,

$$E(V_2|d_2,i^{k_1}) = \begin{cases} \alpha - J_0 - ZNT\ E(X_1|i^{k_1}) & , \text{ if } d_2 = a_2 \\ 0 & , \text{ if } d_2 = r_2 \end{cases} .$$

In the speical case where $k_1 = *_1$, this result is the same as Howard's Equations 2 and 3. The decision rule for d_2 is obtained by maximizing the expectation of V_2 given d_2, k_1, and $I^{k_1} = i^{k_1}$ with respect to d_2. The result is

$$\tilde{d}_2 = \begin{cases} a_2, \text{ if } \alpha - J_0 - ZNT\ E(X_1|i^{k_1}) > 0 \\ r_2, \text{ if } \alpha - J_0 - ZNT\ E(X_1|i^{k_1}) \leq 0 \end{cases}$$

where the choice $\tilde{d}_2 = r_2$ if $\alpha - J_0 - ZNT\ E(X_1|i^{k_1}) = 0$ was made arbitrarily.

The experiment rule, $\tilde{k}_1$, is found by maximizing the expression,

$$\int_{\Omega_{k_1}} f_{I^{k_1}}(\ ) \max_{d_2} E(V_2|d_2,i^{k_1})di^{k_1} - C(k_1)$$

with respect to k_1.

Assuming the hypothesis of Theorem 4 is satisfied, the following upper bound on $C(\tilde{k}_1)$ is obtained,

$$C(\tilde{k}_1) \le E^{\hat{k}_1}_{\max}(V_2|d_1) - E^{*_1}_{\max}(V_2|d_1) = E^{\hat{k}_1}_{\max}(V_2) - E^{*_1}_{\max}(V_2)$$

$$= \int_{\Omega_{X_1}} f_{X_1}(\ ) \max_{d_2} E(V_2|d_2,x_1)dx_1 - \max_{d_2} E(V_2|d_2)$$

$$= \int_0^\infty f_{X_1}(\ ) \left[\max_{d_2} E(V_2|d_2,x_1) - \max_{d_2} E(V_2|d_2) \right] dx_1$$

$$= \begin{cases} \displaystyle\int_{\frac{\alpha-J_0}{ZNT}}^{\infty} [ZNT\ x_1 + J_0 - \alpha]\ f_{X_1}(\ )dx_1 \ , \text{ if } E(X_1) < \frac{\alpha-J_0}{ZNT} \\ \displaystyle\int_0^{\frac{\alpha-J_0}{ZNT}} [\alpha - J_0 - ZNT\ x_1]\ f_{X_1}(\ )dx_1 \ , \text{ if } E(X_1) \ge \frac{\alpha-J_0}{ZNT} \end{cases} .$$

The bound obtained for $E(X_1) < \frac{\alpha-J_0}{ZNT}$ is the same as Howard's Equation 4 which he calls the value of clairvoyance.

Suppose now that $\omega_1 = \{1_1, 2_1, 3_1, \ldots, p_1, \ldots\}$ where p_1, $p = 1,2,\ldots,$ represents placing p devices in operation, noting their times of failure, and averaging those times to obtain an estimate of X_1. The cost, $C(p_1)$, would depend on the particular problem, but usually would be a monotone, increasing, unbounded function of p. In this case, the bound for $C(\tilde{k}_1)$ would make it possible to eliminate from consideration all but a finite number of the experiments in ω_1.

CHAPTER VI

SUMMARY AND CONCLUSIONS

Based on a desire to study a number of practical problems with the same basic characteristics, a generalized, finite stage, discrete time, sequential decision theory problem has been defined. The problem is characterized by a sequence of decisions resulting in unobservable outcomes combined with a choice, between decisions, of experiments which may produce information about previous outcomes. The problem is believed by the author to be original.

A general solution to the problem was given, but unfortunately the procedure involved is frequently formidable. This fact was the motivating force behind efforts to find methods which would reduce the computational effort in a problem solution. Theorems 1 through 4 represent the results obtained in this direction. The examples in Chapters III, IV, and V illustrate applications of the general problem formulation, the general solution procedure, and Theorems 1 through 4.

Originally, the author thought that there might exist a reasonably simple measure of the information in an experiment which could be used as a shortcut to determine the optimal experiment rules and thus reduce the work involved in the solution procedure. Unfortunately, the only measure of information discovered which determines, in a general way whether one experiment is better than another is one like that defined in Chapter II. Evaluating that measure amounts to solving the problem directly and, therefore, affords no reduction in the work required. Theorems 1 and 2 are, in the case of cost free experiments, characterizations of experiments with the least and most information of any of the available experiments respectively. The bounds in Theorems 3 and 4 can be interpreted as the most information which can be contained in any of the available experiments. Although it has not been established rigorously, the implication is that any meaningful measure of information must be defined relative to the particular problem at hand and is not absolute.

In practice, the number of decisions in a problem may be quite large. The solution of such problems can be expected to require the aid of machine computation. Investigation of conditions under which a stationary solution is approached seems to be a reasonable avenue of approach to reduce computational effort when n is large and is a suggestion for future investigation.

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APPENDIX

The purpose of this appendix is to establish Theorems 1 through 4 of Chapter II. The following two Lemmas are developed for that purpose.

Lemma 1

Let $U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})$ denote the maximum expectation of V - C given $\underline{d}_{j-1}$, $\underline{k}_{j-1}$, and $I^{\underline{k}_{j-1}} = i^{\underline{k}_{j-1}}$. Let $\hat{d}_j, \hat{k}_j, \hat{d}_{j+1}, \ldots, \hat{k}_{n-1}, \hat{d}_n$ be the decision and experiment choice rules for $U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})$ with $k_{j-1} = {}^{*}{}_{j-1}$. Define $L_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})$ to be $U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}})$ but with the rules $\hat{d}_j, \hat{k}_j, \hat{d}_{j+1}, \ldots, \hat{k}_{n-1}, \hat{d}_n$. Then the following two properties hold:

$$(1)\quad L_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-2}}, I^{*_{j-1}}}) = U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-2}}, I^{*_{j-1}}})$$

$$(2)\quad L_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}}) \leq U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}}) \quad .$$

The first property is true because of the way L_j is defined. The second property is true because no set of rules can be better than the optimum set.

Lemma 2

Suppose that

$$f_{I^{\underline{k}_{n-1}}|\underline{D}_n,\underline{X}_n} = f_{I^{k_1}|\underline{X}_1} f_{I^{k_2}|\underline{X}_2} \cdots f_{I^{k_{n-1}}|\underline{X}_{n-1}} .$$

Then, given $\underline{d}_{j-1}$, $\underline{k}_{j-1}$, and $I^{\underline{k}_{j-2}} = i^{\underline{k}_{j-2}}$, define

$$\gamma_{j-1}(f_{I^{k_{j-1}}|\underline{X}_{j-1}}(\,|\,)) = f_{I^{k_{j-1}}|\underline{D}_{j-1},I^{\underline{k}_{j-2}}} U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{j-1}}})$$

$$= \int_{\Omega_{\underline{X}_{j-1}}} f_{I^{k_{j-1}}|\underline{X}_{j-1}}(\,|\,) f(\,|\,)_{\underline{X}_{j-1}|\underline{D}_{j-1},I^{\underline{k}_{j-2}}} d\underline{x}_{j-1}$$

$$U_j\left[\frac{f_{I^{k_{j-1}}|\underline{X}_{j-1}}(\,|\,)\; f(\,|\,)_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{j-2}}}}{f_{I^{k_{j-1}}|\underline{D}_{j-1},I^{\underline{k}_{j-2}}}(\,|\,)}\right]$$

where U_j is as defined in Lemma 1.

Also, if k_{j-1} and k'_{j-1} are elements of ω_{j-1} and $i^{k_{j-1}}$ is a sample value of $I^{k_{j-1}}$, let $\hat{d}_j$, $\hat{k}_j$, $\hat{d}_{j+1}$, ..., $\hat{k}_{n-1}$, $\hat{d}_n$ denote the decision and experiment choice rules for $U_j(f(\,|\,)_{\underline{X}_n|\underline{D}_n,I^{\underline{k}_{j-2}},I^{k_{j-1}}})$ evaluated at $i^{k_{j-1}}$ and define $L_{i^{k_{j-1}}}(f_{I^{k'_{j-1}}|\underline{X}_{j-1}}(\,|\,))$ to be $\gamma_{j-1}(f_{I^{k'_{j-1}}|\underline{X}_{j-1}}(\,|\,))$ but with the rules $\hat{d}_j$, $\hat{k}_j$, $\hat{d}_{j+1}$, ..., $\hat{k}_{n-1}$, $\hat{d}_n$. Then the following two properties hold:

$$(1)\quad L_{i^{k_{j-1}}}(f_{I^{k_{j-1}}|\underline{X}_{j-1}}(|)) = \gamma_{j-1}(f_{I^{k_{j-1}}|\underline{X}_{j-1}}(|))$$

$$(2)\quad L_{i^{k_{j-1}}}(f_{I^{k'_{j-1}}|\underline{X}_{j-1}}(|)) \leq \gamma_{j-1}(f_{I^{k'_{j-1}}|\underline{X}_{j-1}}(|)) \quad .$$

The first property is true because of the way $L_{i^{k_j}}$ is defined. The second property is true because no set of rules can be better than the optimum set.

Theorems 1 through 4 can now be established using Lemmas 1 and 2.

Theorem 1

Suppose $\omega_{n-1} \sim \{*_{n-1}\}$ is not empty and let $k_{n-1}\varepsilon\omega_{n-1}$, $k_{n-1} \neq *_{n-1}$. The functional L_n of Lemma 1 can be written

$$L_n(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|)) = \int_{\Omega_{\underline{X}_n}} V(\underline{x}_n, \underline{d}_{n-1}, \hat{d}_n)\, f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|\hat{d}_n)d\underline{x}_n$$

where $\hat{d}_n$ is the decision rule of Lemma 1. Using properties 1 and 2 of Lemma 1,

$$\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(|)\, L_n(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|))di^{k_{n-1}}$$

$$= L_n(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-2}}, I^{*_{n-1}}}(|)) = U_n(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-2}}, I^{*_{n-1}}}(|))$$

$$\leq \int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}}|\underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(|)\, U_n(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{n-1}}}(|))di^{k_{n-1}}$$

where a change in order of integration was made. Thus, for any $k_{n-1} \varepsilon \omega_{n-1}$,

$$\int_{\Omega_{*_{n-1}}} f_{I^{*_{n-1}} | \underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(\,|\,) \; U_n(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-2}}, I^{*_{n-1}}}(\,|\,)) \, di^{*_{n-1}}$$

$$= U_n(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-2}}, I^{*_{n-1}}}(\,|\,))$$

$$\leq \int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}} | \underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(\,|\,) \; U_n(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-1}}}(\,|\,)) \, di^{k_{n-1}}$$

or,

$$\max_{k_{n-1} \varepsilon \omega'_{n-1}} \left[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}} | \underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(\,|\,) \; U_n(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-1}}}(\,|\,)) \, di^{k_{n-1}} \right]$$

$$= \max_{k_{n-1} \varepsilon \omega_{n-1}} \left[\int_{\Omega_{k_{n-1}}} f_{I^{k_{n-1}} | \underline{D}_{n-1}, I^{\underline{k}_{n-2}}}(\,|\,) \; U_n(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-1}}}(\,|\,)) \, di^{k_{n-1}} \right]$$

verifying that ω_{n-1} may be replaced by ω'_{n-1}.

Next suppose $\omega_{n-2} \sim \{*_{n-2}\}$ is not empty and let $k_{n-2} \varepsilon \omega_{n-2}$, $k_{n-2} \neq *_{n-2}$. The functional L_{n-1} of Lemma 1 can be written

$$L_{n-1}(f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-2}}}(\,|\,)) = \int_{\Omega_{\hat{k}_{n-1}}} \int_{\Omega_{\underline{X}_n}} V(\underline{x}_n, \underline{d}_{n-2}, \hat{d}_{n-1}, \hat{d}_n) \; f_{I^{\hat{k}_{n-1}} | \underline{X}_{n-1}}(\,|\,)$$

$$f_{\underline{X}_n | \underline{D}_n, I^{\underline{k}_{n-2}}}(\,|\,\hat{d}_{n-1}, \hat{d}_n) \, d\underline{x}_n \, di^{\hat{k}_{n-1}}$$

where $\hat{d}_{n-1}$, $\hat{k}_{n-1}$, and $\hat{d}_n$ are the decision and experiment choice rules of Lemma 1. Using properties 1 and 2 of Lemma 1 it can be concluded that ω_{n-2} may be replaced by ω'_{n-2}. This line of reasoning may be repeated a finite number of times to obtain the desired conclusion of the theorem.

Theorem 2

Suppose that for some j there is an experiment $k'_j \varepsilon \omega_j$ which satisfies the conditions of the theorem. Given an experiment, $k_j \varepsilon \omega_j$, $k_j \neq k'_j$, define

$$\Gamma(i^{k_j}, \underline{x}_j) = \frac{\int_{\Omega_{k'_j}} z_{k_j}(i^{k_j}, i^{k'_j}) f_{I^{k'_j}|\underline{X}_j}(|)\, di^{k'_j}}{\int_{\Omega_{k'_j}} z_{k_j}(i^{k_j}, \cdot)\, d\cdot} = \frac{f_{I^{k_j}|\underline{X}_j}(|)}{\int_{\Omega_{k'_j}} z_{k_j}(i^{k_j}, \cdot)\, d\cdot} .$$

Let $\gamma_j(f_{I^{k_j}|\underline{X}_j}(|)\ )$ be the functional defined in Lemma 2. Notice from the definition of γ_j that since $z_{k_j}(i^{k_j}, i^{k'_j})$ does not depend on $\underline{x}_n$,

$$\gamma_j\left[\frac{f_{I^{k_j}|\underline{X}_j}(|)}{\int_{\Omega_{k'_j}} z_{k_j}(i^{k_j}, \cdot)\, d\cdot}\right] = \gamma_j(f_{I^{k_j}|\underline{X}_j}(|)) \frac{1}{\int_{\Omega_{k'_j}} z_{k_j}(i^{k_j}, \cdot)\, d\cdot} .$$

Now,

$$\int_{\Omega_{k_j}} f_{I^{k_j}|\underline{D}_j, I^{\underline{k}_{j-1}}}(\,|\,)\; U_{j+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_j}}(\,|\,))\,di^{k_j} = \int_{\Omega_{k_j}} \gamma_j(f_{I^{k_j}|\underline{X}_j}(\,|\,))\,di^{k_j}$$

$$= \int_{\Omega_{k_j}} \gamma_j(\nabla(i^{k_j}, \underline{x}_j)) \int_{\Omega_{k_j'}} z_{k_j}(i^{k_j}, \cdot)d\cdot\; di^{k_j} \quad .$$

Notice that $\nabla(i^{k_j}, \underline{x}_j)$ is the expectation of $f_{I^{k_j'}|\underline{X}_j}$ under the density function,

$$\frac{z_{k_j}(i^{k_j}, i^{k_j'})}{\int_{\Omega_{k_j'}} z_{k_j}(i^{k_j}, \cdot)d\cdot} \quad .$$

The functional $L_{i^{k_j}}$, defined in Lemma 2, can be written

$$L_{i^{k_j}}(f_{I^{k_j'}|\underline{X}_j}(\,|\,))$$

$$= \int_{\Omega_{\hat{k}_{j+1}}} \cdots \int_{\Omega_{\hat{k}_{n-1}}} \int_{\Omega_{\underline{X}_n}} V(\underline{x}_n, \underline{d}_j, \hat{d}_{j+1}, \ldots, \hat{d}_n) f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{j-1}}}(\,|\,\hat{d}_{j+1} \ldots \hat{d}_n)$$

$$f_{I^{k_j'}|\underline{X}_j}(\,|\,)\, f_{I^{\hat{k}_{j+1}}|\underline{X}_{j+1}}(\,|\,) \cdots f_{I^{\hat{k}_{n-1}}|\underline{X}_{n-1}}(\,|\,)\; d\underline{x}_n\, di^{\hat{k}_{n-1}} \cdots di^{\hat{k}_{j+1}} \quad .$$

Using properties 1 and 2 of Lemma 2,

$$\int_{\Omega_{k_j'}} \frac{z_{k_j}(i^{k_j}, i^{k_j'})}{\int_{\Omega_{k_j'}} z_{k_j}(i^{k_j},\cdot)\,d\cdot}\, L_{i^{k_j}}(f_{I^{k_j'}|\underline{X}_j}(|))\,di^{k_j'}$$

$$= L_{i^{k_j}}(\Gamma(i^{k_j},\underline{x}_j)) = \gamma_j(\Gamma(i^{k_j},\underline{x}_j))$$

$$\leq \int_{\Omega_{k_j'}} \frac{z_{k_j}(i^{k_j}, i^{k_j'})}{\int_{\Omega_{k_j'}} z_{k_j}(i^{k_j},\cdot)\,d\cdot}\, \gamma_j(f_{I^{k_j'}|\underline{X}_j}(|))\,di^{k_j'}$$

where a change in order of integration was made. Thus,

$$\int_{\Omega_{k_j}} f_{I^{k_j}|\underline{D}_j, I^{\underline{k}_j-1}}(|)\; U_{j+1}(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_j}})\,di^{k_j}$$

$$\leq \int_{\Omega_{k_j}} \int_{\Omega_{k_j'}} z_{k_j}(i^{k_j}, i^{k_j'})\; \gamma_j(f_{I^{k_j'}|\underline{X}_j}(|))\,di^{k_j'}\;di^{k_j}$$

$$= \int_{\Omega_{k_j'}} f_{I^{k_j'}|\underline{D}_j, I^{\underline{k}_j-2}}(|)\; U_{j+1}(f(|)_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_j-1}, I^{k_j'}})\,di^{k_j'} \quad .$$

Since this result holds for any $k_j \varepsilon \omega_j$, $k_j \neq k_j'$, the desired conclusion is obtained.

Theorem 3

Suppose there is an experiment k_p' which satisfies the conditions of the theorem. Given an experiment, $k_p \varepsilon \omega_p$, $k_p \neq k_p'$ and $k_p \neq *_p$, it can be established, by the same procedure used to argue Theorem 2, that the following inequality holds.

$$\int_{\Omega_{k_p}} f_{I^{k_p}|\underline{D}_p, I^{\underline{k}_{p-1}}}(|)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_p}}(|))\, di^{k_p}$$

$$\leq \int_{\Omega_{k_p'}} f_{I^{k_p'}|\underline{D}_p, I^{\underline{k}_{p-1}}}(|)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{p-1}}, I^{k_p'}}(|))\, di^{k_p'} + C(k_p') \quad .$$

Using the expanded form, this inequality may be rewritten,

$$\sum_{j=1}^{p} \int_{\Omega_{\underline{X}_j}} V_j(\underline{x}_j, \underline{d}_j)\; f_{\underline{X}_j|\underline{D}_p, I^{\underline{k}_{p-1}}}(|)\; d\underline{x}_j$$

$$+ \left[E_{max}^{k_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j, \underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j) \,|\, \underline{d}_p, i^{\underline{k}_{p-1}} \right) - C(k_p) \right] - \sum_{j=1}^{p-1} C(k_j)$$

$$\leq \sum_{j=1}^{p} \int_{\Omega_{\underline{X}_j}} V_j(\underline{x}_j, \underline{d}_j)\; f_{\underline{X}_j|\underline{D}_p, I^{\underline{k}_{p-1}}}(|)\; d\underline{x}_j$$

$$+ \left[E_{max}^{k_p'}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j, \underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j) \,|\, \underline{d}_p, i^{\underline{k}_{p-1}} \right) \right] - \sum_{j=1}^{p-1} C(k_j)$$

or,

$$E_{max}^{k_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) - C(k_p)$$

$$\leq E_{max}^{k_p'}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) \quad .$$

Also, the same procedure used to argue Theorem 1 can be used to establish the following inequalities.

$$\int_{\Omega_{*_p}} f_{I^{*_p}|\underline{D}_p, I^{\underline{k}_{p-1}}}(\,|\,)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{p-1}}, I^{*_p}}(\,|\,))\,di^{*_p}$$

$$\leq \int_{\Omega_{k_p'}} f_{I^{k_p'}|\underline{D}_p, I^{\underline{k}_{p-1}}}(\,|\,)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{p-1}}, I^{k_p'}}(\,|\,))\,di^{k_p'} + C(k_p')$$

and,

$$\int_{\Omega_{*_p}} f_{I^{*_p}|\underline{D}_p, I^{\underline{k}_{p-1}}}(\,|\,)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_{p-1}}, I^{*_p}}(\,|\,))\,di^{*_p}$$

$$\leq \int_{\Omega_{k_p}} f_{I^{k_p}|\underline{D}_p, I^{\underline{k}_{p-1}}}(\,|\,)\; U_{p+1}(f_{\underline{X}_n|\underline{D}_n, I^{\underline{k}_p}}(\,|\,))\,di^{k_p} + C(k_p) \quad .$$

Using the expanded form, these inequalities may be rewritten,

$$E_{max}^{*_p}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

$$\leq E_{max}^{k_p'}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

and,

$$\overset{*}{E}{}^{p}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

$$\leq \overset{k_p}{E}{}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) \quad .$$

Thus,

$$\max_{k_p \varepsilon \omega_p} \left[\overset{k_p}{E}{}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) - C(k_p) \right]$$

$$= \overset{\tilde{k}_p}{E}{}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) - C(\tilde{k}_p)$$

$$\leq \overset{k'_p}{E}{}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

and,

$$\overset{*}{E}{}^{p}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right)$$

$$\leq \overset{\tilde{k}_p}{E}{}_{\max}\left(\sum_{j=p+1}^{n} V_j(\underline{x}_j,\underline{d}_j) - \sum_{j=p+1}^{n-1} C(k_j)\,|\,\underline{d}_p, i^{\underline{k}_{p-1}}\right) \quad .$$

The last two inequalities combined establish the desired result.

Theorem 4

Theorem 4 is a corollary to Theorem 3. Let $\hat{k}_p$ serve as the experiment k'_p in Theorem 3 and let $f_{I^{k_p}|\underline{X}_p}(i^{\hat{k}_p}|i^{\hat{k}_p})$ serve as the function $z_{k_p}(i^{k_p}, i^{\hat{k}_p})$.

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